CHAPTER 1-3: ENVELOPE THEOREM:

Effect of a parameter change on the maximized value

Class discussion

A multiproduct firm has a cost function $C(q)$ and is a price taker in its output markets.

Maximized profit is $\Pi(p) = \max_q \{p \cdot q - C(q)\}$. Suppose that for any $p$ there is a unique maximizer $q^*(p)$.

With data only on profit and prices and outputs, what can be said about the elasticity of profit with respect to the price of commodity $j$?
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Simplistic approach: Assume that the firm does not change its output.

Then $\Pi = \sum_{j=1}^{n} p_j q_j^* - C(q^*)$. Hence $\frac{\partial \Pi}{\partial p_j} = q_j^*(p)$.

Therefore $\varepsilon(\Pi, p_j) = \frac{p_j}{\Pi} \frac{\partial \Pi}{\partial p_j} = \frac{p_j q_j^*(p)}{\Pi(p)}$. 

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A multiproduct firm has a cost function $C(q)$ and is a price taker in its output markets.

Maximized profit is $\Pi(p) = \text{Max}_{q} \{ p \cdot q - C(q) \}$. Suppose that for any $p$ there is a unique maximizer $q^*(p)$.

$$ q^*(p) = \text{arg} \max_{q} \{ p \cdot q - C(q) \}. $$

With data only on profit and prices and outputs, what can be said about the elasticity of profit with respect to the price of commodity $j$?

Simplistic approach: Assume that the firm does not change its output.

Then $\Pi = \sum_{j=1}^{n} p_j q_j^* - C(q^*)$. Hence $\frac{\partial \Pi}{\partial p_j} = q_j^*(p)$.

Therefore $\varepsilon(\Pi, p_j) = \frac{p_j \frac{\partial \Pi}{\partial p_j}}{\Pi} = \frac{p_j q_j^*(p)}{\Pi(p)}$.

Sophisticated response: Take into account the effect of the price change on output.
\[ \Pi(p) = p \cdot q^*(p) - C(q^*(p)) \]

**Example:** \( p^0 = 100, \ C(q) = q^2 \)

Profit, \( pq - q^2 \) is maximized at \( q^*(p) = \frac{1}{2} p \). Then \( q^*(p^0) = 50 \) and so \( \Pi^0 = 2500 \).

**Simplistic answer:**

\[ \frac{\partial \Pi}{\partial p} = q^*(p^0) = 50 \text{ therefore } \varepsilon(\Pi, p) = \frac{p \frac{\partial \Pi}{\partial p}}{\Pi} = \frac{pq^*(p)}{\Pi(p)} = \frac{5000}{2500} = 2. \]
**Example:** \( p^0 = 100, \ C(q) = q^2 \)

Profit, \( pq - q^2 \) is maximized at \( q^* (p) = \frac{1}{2} p \). Then \( q^* (p^0) = 50 \) and so \( \Pi^0 = 2500 \).

**Simplistic answer:**

\[
\frac{\partial \Pi}{\partial p} = q^* (p^0) = 50 \text{ therefore } \frac{\Pi}{\partial p} = \frac{pq^* (p)}{\Pi(p)} = \frac{5000}{2500} = 2.
\]

**Sophisticated answer:**

\[
\Pi(p) = pq^* (p) - C(q^*) = pq^* (p) - (q^*)^2 = p(\frac{1}{2} p) - (\frac{1}{2} p)^2 = \frac{1}{4} p^2.
\]

Therefore \( \frac{\partial \Pi}{\partial p}(p) = \frac{1}{2} p \) and so \( \frac{\partial \Pi}{\partial p}(p^0) = \frac{1}{2} p^0 = 50 \)

Same answer as in the naïve approach.

What is going on here?
**Envelope Theorem I**

Define $F(\alpha) = f(x^*(\alpha), \alpha))$ where $\{x^*(\alpha)\} = \arg \max_{x} f(x, \alpha) \mid x \in X \subseteq \mathbb{R}^n$ and $f \in C^1$.

If $x^*(\alpha)$ is a continuous function then $\frac{dF}{d\alpha} = \frac{\partial f}{\partial \alpha}(x^*(\alpha), \alpha)$.

**Proof:** Consider any $\alpha^1$ and $\alpha^2$ and maximizers $x^*(\alpha^1)$ and $x^*(\alpha^2)$.

Since $x^*(\alpha^1)$ is optimal with parameter $\alpha^1$,

\[ f(x^*(\alpha^1), \alpha^1) \geq f(x^*(\alpha^2), \alpha^1) \]. Then

\[ F(\alpha^2) - F(\alpha^1) = f(x^*(\alpha^2), \alpha^2) - f(x^*(\alpha^1), \alpha^1) \leq f(x^*(\alpha^2), \alpha^2) - f(x^*(\alpha^2), \alpha^1) \]

Similarly, since $x^*(\alpha^2)$ is optimal with parameter $\alpha^2$.

\[ f(x^*(\alpha^1), \alpha^2) \leq f(x^*(\alpha^2), \alpha^2) \]. Then

\[ F(\alpha^2) - F(\alpha^1) = f(x^*(\alpha^2), \alpha^2) - f(x^*(\alpha^1), \alpha^1) \geq f(x^*(\alpha^1), \alpha^2) - f(x^*(\alpha^1), \alpha^1) \]
Then $F(\alpha_2) - F(\alpha_1)$ satisfies

$$f(x^*(\alpha^1), \alpha^2) - f(x^*(\alpha^1), \alpha^1) \leq F(\alpha^2) - F(\alpha^1) \leq f(x^*(\alpha^2), \alpha^2) - f(x^*(\alpha^2), \alpha^1)$$

Since $f$ is continuous, both the left hand and right hand terms approach zero as $\alpha^2 \to \alpha^1$. Therefore the maximized value function (or “value function”) is a continuous function.

Also,

$$\frac{f(x^*(\alpha^1), \alpha^2) - f(x^*(\alpha^1), \alpha^1)}{\alpha^2 - \alpha^1} \leq \frac{F(\alpha_2) - F(\alpha_1)}{\alpha^2 - \alpha^1} \leq \frac{f(x^*(\alpha^2), \alpha^2) - f(x^*(\alpha^2), \alpha^1)}{\alpha^2 - \alpha^1}$$

If $x^*(\alpha)$ is continuous, then taking the limit,

$$\frac{\partial f}{\partial \alpha}(x^*(\alpha^1), \alpha^1) \leq \frac{dF}{d\alpha}(\alpha^1) \leq \frac{\partial f}{\partial \alpha}(x^*(\alpha^1), \alpha^1).$$

Q.E.D.
**Continuity of the maximizer** $x^*(\alpha)$

**Proposition 1.3-2: Continuity of the optimum**¹

If $X$ is compact, $f$ is continuous and $x^*(\alpha)$ is the unique maximizer for $\text{Max}_{x \in X} \{f(x, \alpha)\}$, then $x^*(\alpha)$ is continuous.

**Proof:** Consider the sequence $\{\alpha'\}$ with limit $\alpha^0$ and sequence of optimal choices $\{x^*(\alpha')\}$. Since $X$ is bounded, there must be some convergent subsequence $\{x^*(\alpha^s)\}$. Let the limit point be $\hat{x}$. This is depicted opposite.

¹ This is a special case of the Theorem of the Maximum. See EM Appendix C.

[Diagram showing the continuity of the optimum with points $x^*(\alpha')$ and $x^*(\alpha^0)$]
Suppose that there is a discontinuity at $\alpha^0$ so that the limit of the convergent subsequence $\hat{x} \neq x^*(\alpha^0)$. Since $x^*(\alpha^0)$ solves $\max_{x \in X} \{f(x, \alpha^0)\}$ it follows that for some $\varepsilon > 0$,

$$f(x^0, \alpha^0) < f(x^*(\alpha^0), \alpha^0) - \varepsilon.$$ 

The sequence $\{x^*(\alpha^s)\}$ converges to $\hat{x}$. Thus for all $s$ sufficiently large,

$$f(x^*(\alpha^s), \alpha^s) < f(x^*(\alpha^0), \alpha^0) - \frac{1}{2}\varepsilon$$

But this is impossible since $x^*(\alpha^0) \in X$ and by hypothesis $x^*(\alpha^s)$ solves $\max_{x \in X} \{f(x, \alpha^s)\}$.

Q.E.D.
**Envelope Theorem II**

Define \( F(\alpha) = \max_x \{ f(x, \alpha) \mid x \geq 0, \ h(x, \alpha) \geq 0 \} \)

If \( x^*(\alpha) \) and \( \lambda^*(\alpha) \) are continuously differentiable functions, then

\[
\frac{dF}{d\alpha} = \frac{\partial L}{\partial \alpha}(x^*(\alpha), \lambda^*(\alpha), \alpha).
\]

Class exercise to prove this.

What guarantees that \( x^*(\alpha) \) and \( \lambda^*(\alpha) \) are continuously differentiable functions?
**Implicit function theorem**

Suppose that \( x(\alpha^0) \) is a solution to \( g_i(x, \alpha) = 0, \ i = 1, \ldots, n \) where \( g_i \in C^1 \). Suppose also that the Jacobean matrix \( J(x) = \begin{bmatrix} \frac{\partial g_i}{\partial x_j}(x) \end{bmatrix} \) has an inverse at \( x^0 \).

(For those who remember, this is the case if the determinant of the matrix is non-zero.)

Then for some delta-neighborhood of \( \alpha^0 \), the equation system implicitly defines the continuously differentiable functions \( x_j(\alpha), \ j = 1, \ldots, n \)

Understanding the result…. Consider the linear approximations.
**Implicit function theorem**

Suppose that \( x(\alpha^0) \) is a solution to \( g_i(x, \alpha) = 0, \ i = 1, \ldots, n \) where \( g_i \in C^1 \). Suppose also that the Jacobian matrix \( J(x, \alpha) = \begin{bmatrix} \frac{\partial g_i}{\partial x_j}(x, \alpha) \end{bmatrix} \) has an inverse at \( (x^0, \alpha^0) \).

Then for some delta-neighborhood of \( \alpha^0 \), the equation system implicitly defines the continuously differentiable functions \( x_j(\alpha), \ j = 1, \ldots, n \)

Understanding the result:

Linear approximations: 
\[
\frac{\partial g_i}{\partial x}(x^0, \alpha^0) \cdot (x - x^0) + \frac{\partial g_i}{\partial \alpha}(x^0, \alpha^0)(\alpha - \alpha^0) = 0, \ i = 1, \ldots, n
\]

In matrix notation 
\[
J(x^0, \alpha^0)(x - x^0) + \frac{\partial g}{\partial \alpha}(x^0, \alpha^0)(\alpha - \alpha^0) = 0.
\]

Then 
\[
x - x^0 = -J^{-1} \frac{\partial g}{\partial \alpha}(x^0, \alpha^0)(\alpha - \alpha^0)
\]