CHAPTER 12 SLIDES Sections 4-6

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DESIGN OF AN EFFICIENT MECHANISM

The private benefit to agent \( i \in I \), \( B_i(\theta_i, q) \), depends on his type \( \theta_i \in \Theta_i \subset \mathbb{R} \) and an allocation rule \( q = (q_1, \ldots, q_n) \in Q \). We assume that for agent \( i \) the lowest type is \( \alpha_i \) and the highest is \( \beta_i \).

Preferences are quasi-linear so, if the agent makes a payment of \( r_i \), his payoff is

\[
    u_i = B_i(\theta_i, q) - r_i.
\]

The social surplus is the sum of the private benefits less the cost of the allocation

\[
    S(\theta, q) = \sum_{i \in I} B_i(\theta_i, q) - C(q).
\]

An allocation rule \( q^*(\theta) \) is efficient if

\[
    q^*(\theta) \in \arg\max_{q \in Q} \{ S(\theta, q) \}.
\]
We can rewrite social surplus as follows.

\[ S(\theta, q) = \sum_{i \in \mathcal{I}} (B_i(\theta, q) - r_i) + \sum_{i \in \mathcal{I}} r_i - C(q) \]

\[ = \sum_{i \in \mathcal{I}} u_i(\theta, q) + \left[ \sum_{i \in \mathcal{I}} r_i - C(q) \right] \]

\[ \text{sum of agents' payoffs} \quad + \quad \text{designer profit} \]

Let \( U^D \) be the designer’s profit. Then

\[ U^D = \sum_{i \in \mathcal{I}} r_i - C(q) = S(\theta, q) - \sum_{i \in \mathcal{I}} u_i(\theta, q). \]

A key question that we will address is whether the designer may have to incur an expected loss in order to implement an efficient allocation.
Example 1: Allocation of a single private good (auction)

The allocation rule allocates the single item to agent $i$ with probability $q_i$ and the private benefit is $B_i(\theta, q) = \theta_i q_i$. The social value of the item is then

$$S(\theta, q) = \sum_{i \in J} \theta_i q_i \quad \text{where} \quad Q = \{q \mid q \geq 0 \quad \text{and} \quad \sum_{i \in J} q_i = 1\}.$$
**Example 2: Allocation of a single public good**

A good is “public” if consumption is non-rivalrous.

Each agent places some value on the public good. For social efficiency the good should be produced if and only if the sum of the benefits exceeds the cost.

The total value of the public good is \( \sum_{i \in I} \theta_i \). The cost of the public good is \( k \).

Let \( q(\theta) \) be the probability that the public good is produced given that agent \( i \in I \) has a value \( \theta_i \). The social surplus is then

\[
S(\theta, q(\theta)) = q(\theta)(\sum_{i \in I} \theta_i - k)
\]

For an efficient mechanism the public good should be produced if and only if the sum of the values exceeds the cost. Thus the efficient allocation rule is

\[
q^*(\theta) = \begin{cases} 
0, & \sum_{i \in I} \theta_i < k \\
1, & \sum_{i \in I} \theta_i \geq k 
\end{cases}
\] (1)
V-C-G Mechanism

Key idea: provide an incentive for each agent to reveal his true type by aligning private and social incentives.

As we shall see, the key to achieving efficiency is to align private and social incentives. Suppose that $q^*(\theta)$ maximizes social surplus. That is,

$$q^*(\theta) \in \arg\max_q \{ \sum_{j=1}^n B_j(\theta_j, q) - C(q) \} \text{ for all } \theta \in \Theta.$$ 

It follows immediately that

$$\theta \in \arg\max_q \{ \sum_{j=1}^n B_j(\theta_j, q^*(x)) - C(q^*(x)) \} \text{ for all } x \text{ and } \theta \in \Theta.$$ 

Setting $x_{-i} = \theta_{-i}$

$$\theta_i \in \arg\max_{x_i \in \Theta_i} \{ \sum_{j=1}^n B_j(\theta_j, q^*(x_i, \theta_{-i})) - C(q^*(x_i, \theta_{-i})) \} \text{ for all } \theta \in \Theta.$$
It is helpful to focus on agent $i$ and rewrite this as follows:

$$
\theta_i \in \arg \max_{x_j \in \Theta_j} \{ B_i(\theta_i, q^*(x_i, \theta_{-i})) + t_i(x_i, \theta_{-i}) \} \text{ for all } \theta \in \Theta
$$

where

$$
t_i(x_i, \theta_{-i}) \equiv \sum_{\substack{j \in J \setminus \{i\}}} B_j(\theta_j, q^*(x_i, \theta_{-i})) - C(q^*(x_i, \theta_{-i}))
$$

Simply changing notation,

$$
\theta_i \in \arg \max_{x_j \in \Theta_j} \{ B_i(\theta_i, q^*(x_i, x_{-i})) + t_i(x_i, x_{-i}) \} \text{ for all } \theta \in \Theta_i, x_{-i} \in \Theta_{-i}
$$

where

$$
t_i(x_i, x_{-i}) \equiv \sum_{\substack{j \in J \setminus \{i\}}} B_j(\theta_j, q^*(x_i, x_{-i})) - C(q^*(x_i, x_{-i}))
$$
With these preliminaries we are ready to consider incentives. Suppose that the designer invites each agent \( i \in I \) to announce some \( x_i \in \Theta_i \). If so the designer will use the allocation rule \( q^*(x) \). Once all the agents have made their announcements, agent \( i \), will receive the transfer payment \( t_i(x) \). The agent’s private benefit is \( B_i(\theta_i, q^*(x_i, x_{-i})) \). Therefore his payoff is

\[
u_i(\theta_i, x_i, x_{-i}) = B_i(\theta_i, q^*(x_i, x_{-i})) + t_i(x)\]

\[
= B_i(\theta_i, q^*(x_i, x_{-i})) + \sum_{j \in I, j \neq i} B_j(x_j, q^*(x_i, x_{-j})) - C(q^*(x_i, x_{-i})).
\]

We have just argued that for all \( x_{-i} \in \Theta_{-i} \) the right hand side is maximized at \( x_i = \theta_i \). Therefore, regardless of the announcements of the other agents, announcing \( x_i = \theta_i \) is optimal for agent \( i \). That is, revealing the truth is a dominant strategy for agent \( i \). Since our argument holds for every agent, it follows that it is a dominant strategy for all agents to announce their private parameters. If all agents play their dominant strategies it follows that allocation is \( q^*(\theta) \) so efficiency is achieved.
The equilibrium payoff of agent is therefore

\[ U_i(\theta) = u_i(\theta_i, \theta_i, \theta_{-i}) = B_i(\theta_i, q^*(\theta_i, \theta_{-i})) + t_i(\theta_i, \theta_{-i}) \]

\[ = B_i(\theta_i, q^*(\theta_i, \theta_{-i})) + \sum_{j \in \mathbb{J}} B_j (x_j, q^*(\theta_i, \theta_{-i})) - C(q^*(\theta_i, \theta_{-i})) \}

Thus efficiency is achieved by making each agent’s equilibrium payoff is equal to the entire social surplus. However there is a cheaper alternative. Maximized social surplus is an increasing function. It follows that the minimum payoff to agent \( i \) is \( S^*(\alpha_i, \theta_{-i}) \). Consider then the alternative transfer \( t_i^*(x) = t_i(x) - S_i(\alpha_i, x_{-i}) \). Then

\[ U_i^*(\theta) = S^*(\theta_i, \theta_{-i}) - S^*(\alpha_i, \theta_{-i}) \]

Note that the participation constraint is now always binding when the agent’s \( i \)’s type is \( \alpha_i \). Thus \( t_i^*(x) \) is the smallest transfer that is incentive compatible.
Finally note that the incentive scheme aligns agent $i$’s payoff with his contribution to social surplus over and above the minimum contribution. Following Ostroy we will henceforth refer to this as agent $i$’s marginal contribution to social surplus. Providing a transfer equal to agent $i$’s marginal contribution to social surplus gives agent $i$ the incentive to reveal his type.

Our results are summarized in the following proposition.

**Proposition 12.4-1: V-C-G Mechanism**

The efficient allocation $q^*(\theta)$ can be achieved as a dominant strategy equilibrium if, based on the vector of announced values $x$, the allocation is $q^*(x)$ and agent $i$, $i \in \mathcal{I}$ is given a subsidy (possibly negative)

$$t_i^*(x_i, x_{-i}) = \sum_{j \in \mathcal{J}} B(x_j, q^*(x_i, x_{-i})) - S_i^*(\alpha_i, q^*(\alpha_i, x_{-i})).$$

With these payments agent $i$’s equilibrium payoff is his marginal contribution to social surplus $S_i^*(\theta_i, x_{-i}) - S_i^*(\alpha_i, x_{-i})$. Thus the participation constraint is binding for the lowest type.
To complete the analysis we solve for the equilibrium expected payoffs $V_i(\theta_i)$ of each agent.

$$V_i(\theta_i) = \mathbb{E}_{\theta_{-i}} \{ S_i^* (\theta_i, \theta_{-i}) - S_i^* (\alpha_i, \theta_{-i}) \}$$

Then $V_i(\alpha_i) = 0$. Also, $S_i^* (\theta_i, \theta_{-i}) = S(\theta, q^*(\theta)) = \text{Max}_q \{ B_i(\theta_i, q) + \sum_{j \neq i} B_j(\theta_j, q) - C(q) \}.$

By the Envelope Theorem,

$$\frac{\partial S_i^*}{\partial \theta_i} (\theta_i, \theta_{-i}) = \frac{\partial B_i}{\partial \theta_i} (\theta_i, q^*(\theta)).$$

Therefore

$$V_i'(\theta_i) = \mathbb{E} \{ \frac{\partial}{\partial \theta_i} [ S_i^*(\theta_i, \theta_{-i}) - S_i^*(\alpha_i, \theta_{-i})] \}$$

$$= \mathbb{E} \{ \frac{\partial B_i}{\partial \theta_i} (\theta_i, q^*(\theta)) \}$$
We therefore have the following result.

**Proposition 12.4-2: Equilibrium expected payoff to the agents using the V-C-G Mechanism**

Using the V-C-G mechanism to implement an efficient allocation, the equilibrium expected payoff to agent $i \in \mathcal{I}$, $V_i(\theta_i)$, satisfies

(i) $V_i(\alpha_i) = 0$ and (ii) $V_i'(\theta_i) = E\{\frac{\partial B_i}{\partial \theta_i}(\theta_i, q^*(\theta))\}$
Application: Allocation of a single indivisible good with continuous types

2 agents with values $\theta_1, \theta_2$ continuously distributed on $\Theta = [0,1]$.

An allocation rule is a probability vector $q^*(\theta)$. For efficiency the good is allocated to the agent with the highest value. Then

$$B_1(\theta_1, q^*(\theta)) = \begin{cases} 0, & \theta_1 < \theta_2 \\ \theta_1, & \theta_1 \geq \theta_2 \end{cases} \quad \text{and} \quad B_2(\theta_2, q^*(\theta)) = \begin{cases} 0, & \theta_2 < \theta_1 \\ \theta_2, & \theta_2 \geq \theta_1 \end{cases}$$

Also, if agent 1 has a value of zero, maximized surplus is $\theta_2$. Therefore $S_1^*(0, \theta_2) = \theta_2$.

If the agents reveal their true types, the transfer to agent 1 is therefore

$$t_1(\theta_1, \theta_2) = B_2(\theta_2, q^*(\theta)) - S_1(0, \theta_2) = \begin{cases} 0, & \theta_1 < \theta_2 \\ -\theta_2, & \theta_1 \geq \theta_2 \end{cases}.$$ 

If the agents announce $x = (x_1, x_2)$ the transfer is

$$t_1(x_1, x_2) = B_2(x_2, q^*(x)) - S_1(0, x_2) = \begin{cases} 0, & x_1 < x_2 \\ -x_2, & x_1 \geq x_2 \end{cases}.$$ 

1 Since values are equal with zero probability we ignore ties in the exposition.
Recap: \[ t_1(x_1, x_2) = B_2(x_2, q^*(x)) - S_1(0, x_2) = \begin{cases} 0, & x_1 < x_2 \\ -x_2, & x_1 > x_2 \end{cases} \]

If the announced values are \( x = (x_1, x_2) \), the post transfer payoff to agent 1 is therefore

\[ U_1(\theta_1, x) = B_1(\theta_1, q^*(x)) + t_1(x) = \begin{cases} 0, & x_1 < x_2 \\ \theta_1 - x_2, & x_1 > x_2 \end{cases} \]

Case (i) \( x_2 < \theta_1 \): Agent 1 always wants to be allocated the item so maximizes his payoff by choosing \( x_1 \geq \theta_1 \).

Case (ii) \( x_2 > \theta_1 \): Agent 1 never wants to be allocated the item so maximizes his payoff by choosing \( x_1 \leq \theta_1 \).

Agent 1 can satisfy both inequalities by choosing \( x_1 = \theta_1 \). Thus his best response is to announce his true value.

Note that the efficient mechanism can be implemented as a second price auction. If agent 1’s announced value is the highest, he wins the item and pays the second announced value.
Public goods example

Each agent’s value $\theta_i \in \Theta = \{2, 4\}$. The cost of the public good is $k = 5$. Values are independently distributed. A value is high with probability $\pi$.

Note that $2 + 2 < k < 2 + 4$.

Then it is efficient to produce the public good if at least one value is high.

<table>
<thead>
<tr>
<th>Agent 1</th>
<th>$f_{st}$</th>
<th>Agent 2</th>
<th>$B^*_{st}$</th>
<th>Agent 2</th>
<th>$r_{st}$</th>
<th>Agent 2</th>
<th>$\pi(1-\pi)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$\pi(1-\pi)$</td>
</tr>
<tr>
<td>1</td>
<td>$(1-\pi)^2$</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>2</td>
<td>$\pi^2$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$\pi(1-\pi)$</td>
<td>2</td>
<td>$\pi$</td>
<td>2</td>
<td>$\pi$</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

Joint probability distribution | Benefits to agent 1 | Payments by agent 1 | Binding constraint

Expected payment by agent 1 $E\{r_i\} = (1-\pi)^2r_{11} + (1-\pi)\pi r_{12} + \pi(1-\pi)r_{21} + \pi^2 r_{22}$.

Expected cost of public good: $(1-(1-\pi)^2)k$.

Suppose that $\pi = 1/3$

The sum of the expected payments by the two agents $= 2[\frac{2}{9}2 + \frac{2}{9}4 + \frac{1}{9}2] = \frac{28}{9}$.

The expected cost $= \frac{5}{9}5 = \frac{25}{9}$.

Thus the efficient outcome can be implemented with a direct revelation mechanism that is profitable.

Exercise: Show that the designer has a loss if $\pi = 3/4$. 

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## Dominant strategy (Vickrey-Clarke-Groves) mechanism

<table>
<thead>
<tr>
<th>Agent 1</th>
<th>Agent 2</th>
<th>Agent 2</th>
<th>Agent 2</th>
<th>Agent 2</th>
<th>Agent 2</th>
<th>Agent 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$B^*_s$</td>
<td>$S^*_s$</td>
<td>$S^*_t$</td>
<td>$N^*_s$</td>
<td>$r^*_s$</td>
<td>$\theta_1$</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

Benefits to agent 1

Total surplus

Total surplus if agent 1’s value is lowest

Marginal surplus

Choose payments so that $B^*_s - r^*_s = N^*_s$

**Case (i) $\theta_1 = 2$**

<table>
<thead>
<tr>
<th>Agent 1</th>
<th>Agent 2</th>
<th>Agent 2</th>
<th>Agent 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$B(2, x)$</td>
<td>$r(x)$</td>
<td>$u_1(2, x)$</td>
</tr>
<tr>
<td>$x_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x_2$</td>
<td>2</td>
<td>3</td>
<td>-1</td>
</tr>
</tbody>
</table>

**Case (ii) $\theta_1 = 4$**

<table>
<thead>
<tr>
<th>Agent 1</th>
<th>Agent 2</th>
<th>Agent 2</th>
<th>Agent 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$B(4, x)$</td>
<td>$r(x)$</td>
<td>$u_1(4, x)$</td>
</tr>
<tr>
<td>$x_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x_2$</td>
<td>4</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

In each case, truth telling by agent 1 is a dominant strategy.
Note that the payment matrix for agent 1 from the two mechanisms differ. Given symmetry, payments by agent 2 are as shown in the second column. The third column shows the payment sum and also.

\[
\begin{array}{c|c|c}
\text{Agent 1} & \text{Agent 2} & \text{Agent 2} \\
\hline
2 & 2 & 2 \\
4 & 4 & 2 \\
\hline
\end{array}
\]

Agent 1’s payments in Bayesian mechanism

\[
\begin{array}{c|c|c}
\text{Agent 1} & \text{Agent 2} & \text{Agent 2} \\
\hline
2 & 2 & 2 \\
4 & 4 & 2 \\
\hline
\end{array}
\]

Agent 1’s payments in V-C-G mechanism

\[
\begin{array}{c|c|c}
\text{Agent 1} & \text{Agent 2} & \text{Agent 2} \\
\hline
2 & 2 & 2 \\
4 & 4 & 2 \\
\hline
\end{array}
\]

Agent 2’s payments in Bayesian mechanism

\[
\begin{array}{c|c|c}
\text{Agent 1} & \text{Agent 2} & \text{Agent 2} \\
\hline
2 & 2 & 2 \\
4 & 4 & 2 \\
\hline
\end{array}
\]

Agent 2’s payments in V-C-G mechanism

\[
\begin{array}{c|c|c}
\text{Agent 1} & \text{Agent 2} & \text{Agent 2} \\
\hline
2 & 2 & 2 \\
4 & 4 & 2 \\
\hline
\end{array}
\]

Total payments in Bayesian mechanism

\[
\begin{array}{c|c|c}
\text{Agent 1} & \text{Agent 2} & \text{Agent 2} \\
\hline
2 & 2 & 2 \\
4 & 4 & 2 \\
\hline
\end{array}
\]

Total payments in V-C-G mechanism

\[
\begin{array}{c|c|c}
\text{Agent 1} & \text{Agent 2} & \text{Agent 2} \\
\hline
2 & 2 & 2 \\
4 & 4 & 2 \\
\hline
\end{array}
\]

Total payments in V-C-G mechanism

Note that the V-C-G mechanism breaks even in three of the four possible realizations and generates a loss in the fourth. The Bayesian mechanism generates a profit of 1 with

\[
\text{Probability } 2\pi(1-\pi) \text{ and a loss of 1 with probability } \pi^2.
\]

Expected profit is therefore \( E\{U^D\} = 2\pi(1-\pi) - \pi^2 = \pi(2-3\pi). \)

Thus the Bayesian mechanism generates an expected profit if \( \pi < 2/3. \)

As we shall see, with continuously distributed types the expected profit is always negative.
Efficient public good provision

Agent $i$, $i \in \mathcal{I}$, has a private value $\theta_i$ for the public good. The cost of the public good is $k$. Let $q(\theta)$ be the probability that the public good is produced given that agent $i \in \mathcal{I}$ has a value $\theta_i$. The social surplus is then

$$S(\theta, q(\theta)) = q(\theta) \left( \sum_{i \in \mathcal{I}} \theta_i - k \right).$$

For an efficient mechanism the public good should be produced if and only of the sum of the values exceeds the cost. Thus the efficient allocation rule is

$$q^*(\theta) = \begin{cases} 
0, & \sum_{i \in \mathcal{I}} \theta_i < k \\
1, & \sum_{i \in \mathcal{I}} \theta_i \geq k 
\end{cases}$$

(2)

If $\sum_{i \in \mathcal{I}} \alpha_i > k$ it is always optimal to produce the public good and if $\sum_{i \in \mathcal{I}} \beta_i < k$ it is never optimal to produce the public good. We rule out these uninteresting cases by assuming that

$$\sum_{j \in \mathcal{I}} \alpha_j < k < \sum_{j \in \mathcal{I}} \beta_j.$$ 

(3)
the efficient allocation rule is
\[ q^*(\theta) = \arg \max_{q \in [0,1]} \{q(\sum_{j \in J} \theta_j - k)\}. \] (4)

The resulting maximized social surplus can be written as follows
\[ S^*_i(\theta, -\theta_i) = q^*(\theta)\theta_i + [q^*(\theta)(\sum_{j \in J, j \neq i} \theta_j - k)]. \] (5)

Then offer the following transfer to agent \(i\) if the announced types are \(\theta = (\theta_1, \ldots, \theta_n)\)
\[ t_i(\theta) = q^*(\theta)(\sum_{j \in J, j \neq i} \theta_j - k) - S^*_i(\alpha_i, -\theta_i). \]

So \[ t_i(x) = q^*(x)(\sum_{j \in J, j \neq i} x_j - k) - S^*_i(\alpha_i, -\theta_i) = q^*(x_i, -\theta_i)(\sum_{j \in J, j \neq i} x_j - k) - q^*(\alpha_i, -\theta_i)[\alpha_i + \sum_{j \in J, j \neq i} x_j - k] \]

Note: No need to provide an incentive payment if (i) \(q^*(\theta, -\theta_i) = 0\) or (ii) \(q^*(\alpha_i, -\theta_i) = 1\)
V-C-G mechanism yields a loss to the designer

Suppose that \( \theta = (\theta_1, \ldots, \theta_i) \) is large enough so that \( S_i^*(\theta, \theta_{-i}) = S(\theta, q^*(\theta)) > 0 \). Then

\[
S_i^*(\theta, \theta_{-i}) = \max\{0, \theta_i + \sum_{j \neq i} j \in \mathcal{J} \theta_j - k\} = \theta_i + \sum_{j \neq i} j \in \mathcal{J} \theta_j - k
\]

Case (i): \( S_i^*(\alpha, \theta_{-i}) > 0, \quad i \in \mathcal{I} \)

Then

\[
S_i^*(\alpha, \theta_{-i}) = \alpha_i + \sum_{j \neq i} j \in \mathcal{J} \theta_j - k, \quad i \in \mathcal{I}
\]

In the V-C-G mechanism the equilibrium payoff is the agent’s marginal contribution to social surplus

\[
U_i(\theta, \theta) = B_i(\theta, q^*(\theta)) + t_i(\theta) = S_i^*(\theta, \theta_{-i}) - S_i^*(\alpha, \theta_{-i}).
\]

Hence

\[
U_i(\theta, \theta) = \theta_i - \alpha_i.
\]
Recap: \( U_i(\theta_i, \theta) = \theta_i - \alpha_i \) for each agent

Summing over agents,

\[
\sum_{i \in \mathcal{I}} U_i(\theta_i, \theta) = \sum_{i \in \mathcal{I}} \theta_i - \sum_{i \in \mathcal{I}} \alpha_i
\]

By hypothesis, \( \sum_{i \in \mathcal{I}} \alpha_i < k \). Therefore

\[
\sum_{i \in \mathcal{I}} U_i(\theta_i, \theta) > \sum_{i \in \mathcal{I}} \theta_i - k = S_i^*(\theta, \theta_{-i})
\]

Thus the total payoff to the agents exceeds the total surplus.
Case (ii): For $J$ of the agents $S^*_i(\alpha_i, \theta_{-i}) = 0$.

Re-label the agents so that it is the first $J$ for whom this equality holds. Buyer $i$’s equilibrium payoff is his marginal contribution to social surplus

$$U^e_i = S^*_i(\theta, \theta_{-i}) - S^*_i(\alpha_i, \theta_{-i}) = S^*_i(\theta, \theta_{-i}), \text{ } i = 1,\ldots,J$$

Arguing as above, for all the other agents, $U^e_i = \theta - \alpha_i$. Summing over agents,

$$\sum_{i \in J} U^e_i = JS^*_i(\theta, \theta_{-i}) + \sum_{i \in J} (\theta_i - \alpha_i) > JS^*_i(\theta, \theta_{-i}) \quad (6)$$

Thus in both cases the sum of the payoffs exceeds total surplus. Since the total surplus is divided among the agents and the designer it follows that the payoff to the designer is negative. We therefore have the following result.

**Proposition 12.4-3: Designer payoff with efficient public good provision using the V-C-G mechanism**

In the V-C-G mechanism the equilibrium payoff of the designer is zero if the public good is not produced and is strictly negative if the public good is produced.
General efficient mechanisms

Assumption: types independently and continuously distributed

Incentive compatibility

Follow the approach for auctions (direct revelation):

Equilibrium allocation of the mechanism  \( q^*(\theta) \equiv q_i^*(\theta_i, \theta_{-i}) \).

Equilibrium payment by agent \( i: r_i(\theta) \). Define \( \bar{r}_i(\theta_i) = E\{r_i(\theta_i, \theta_{-i})\} \)

Taking the expectation over the other agents,

\[
 u_i(\theta_i, x_i) = E\{B(\theta_i, q^*_i(x_i, \theta_{-i}))\} - E\{r_i(\theta_i, \theta_{-i})\} \equiv E\{B(\theta_i, q^*_i(x_i, \theta_{-i}))\} - \bar{r}_i(\theta_i).
\]

For the mechanism to be incentive compatible, agent \( i \) cannot gain by announcing any other value.
That is,

\[
 u_i(\theta_i, x_i) = E\{B(\theta_i, q^*_i(x_i, \theta_{-i}))\} - \bar{r}_i(x_i), \ x_i \in \Theta_i
\]

\[
 \leq E\{B(\theta_i, q^*_i(\theta_i, \theta_{-i}))\} - \bar{r}_i(\theta_i) \equiv u_i(\theta_i, \theta_i).
\]
Arguing exactly as in the auctions analysis, a necessary and sufficient condition for incentive compatibility is that \( E\{B_i(\theta_i, q^*(\theta))\} \) is an increasing function. **(Is this true?)** Let \( V_i(\theta_i) \) be the equilibrium payoff. Then

\[
V_i(\theta_i) = E\{u_i(\theta_i, x_i)\} = E\{\text{Max}_{x_i} \{B(\theta_i, q^*_i(x_i, \theta_{-i})) - \bar{r}_i(x_i)\}\}, \ x_i \in \Theta_i
\]

Appealing to the Envelope Theorem the equilibrium marginal payoff is

\[
V'_i(\theta_i) = E\left\{ \frac{\partial B_i}{\partial \theta_i}(\theta_i, q^*_i(\theta)) \right\}. \quad (7)
\]

For the mechanism to be non-coercive, \( V_i(\alpha_i) \geq 0 \). To maximize revenue \( V_i(\alpha_i) = 0, \ i \in \mathcal{I} \).

Also, from (7). \( V'_i(\theta_i) = E\left\{ \frac{\partial B_i}{\partial \theta_i}(\theta_i, q^*_i(\theta)) \right\} \). Appealing to Proposition 12.4-2 it follows that agent \( i \)'s equilibrium payoff in the revenue maximizing mechanism is exactly the same as in the V-C-G mechanism.

**Proposition 12.5-1: Revenue-maximizing efficient mechanism**

If values are independently and continuously distributed, the V-C-G mechanism is a revenue maximizing mechanism and so the designer has an expected loss.
**Blocking agents**

Recall from the previous section that the minimum equilibrium payoffs for an efficient V-C-G mechanism are the marginal contributions to social surplus.

\[ u_i(\theta_i, \theta_{-i}) = S^*_i(\theta_i, \theta_{-i}) - S^*_i(\alpha_i, \theta_{-i}) \]  
\[ i = 1, 2, \ldots, n \]

(8)

We have just argued that the V-C-G mechanism yields the agents the lowest incentive compatible returns. Thus any efficient mechanism yields payoffs that are bounded from below by (8).

Suppose that for agent \( i \) it is never efficient to produce the public good when agent \( i \)'s type is \( \alpha_i \), that is

\[ \alpha_i + \sum_{j \in J, j \neq i} \theta_j < k, \quad \forall \theta_{-i} \in \Theta_{-i} \]

Then agent \( i \) can unilaterally block production by announcing that his type is \( \alpha_i \). We call any such agent a blocking agent.
**Definition: Blocking agent**

Agent $i$ is blocking if there are never any social gains to undertaking the project when agent $i$’s value is sufficiently low. Formally, for all $\theta_{-i} \in \Theta_{-i}$, $\alpha_i + \sum_{j \in \mathcal{J}} \theta_j \leq k$ and so $S_i^*(\alpha_i, \theta_{-i}) = 0$

From Proposition 12.4-2 if there are $J$ blocking agents

$$\sum_{i \in \mathcal{J}} u_i(\theta_i, \theta) = JS_i^*(\theta_i, \theta_{-i}) + \sum_{i \in \mathcal{J}, i > J} (\theta_i - \alpha_i) > JS_i^*(\theta_i, \theta_{-i}) \quad (9)$$
Efficient production and exchange with private values and costs

Key ideas: value as cost saving, blocking agent, impossibility theorem, profitable mechanisms with two or more buyers

In the analysis of auctions we showed that if there are \( n \) buyers and a single seller with a known cost of production (or reservation value), it is always possible to design a mechanism that assigns the item to the highest value buyer (and so is efficient) and also generates a profit to the auction designer. Moreover the profit maximizing efficient mechanism is a sealed second-price auction with a reserve price equal to the seller’s cost. But there is an important caveat. The auction is efficient because the seller’s cost is public information. As we shall see, we can apply the results of the public goods model to show that there may exist no such self-supporting auction when it is necessary to elicit the seller’s value as well.

We begin by examining the case in which a commodity is to be produced by a single seller and traded to a single buyer. Agent 1, the buyer, has a value \( \theta_1 \in \Theta_1 = [\alpha_1, \beta_1] \). Agent 2, the seller, can produce the commodity at a cost of \( c \in C = [\gamma, \kappa] \). We assume that \( \theta_1 \) and \( c \) are independently and continuously distributed. Both value and cost are private information.
Efficient allocation = no trade

Efficient allocation = always trade
Buyer is blocking

\[ \alpha_1 < \gamma, \beta_1 > \kappa \]

Seller is blocking

\[ \alpha_1 > \gamma, \beta_1 \leq \kappa \]
Neither agent is blocking

Both are blocking

$$\alpha_1 < \gamma, \quad \beta_1 < \kappa$$

$$\alpha_1 > \gamma, \quad \beta_1 > \kappa$$

Neither agent is blocking
The social gain to the exchange is $\theta_1 - c$. Since both value and cost are private information, just as in the case of the public good, it is necessary to elicit a linear combination of two private signals. It proves helpful to focus not on the seller’s cost but the difference between the seller’s cost and the maximum possible cost $\theta_2 = \kappa - c$. We will refer to this as the cost savings.

Define $\Theta_2$ to be the set of possible cost savings, that is $\Theta_2 = [\alpha_2, \beta_2] = [0, \kappa - \gamma]$. We can then write maximized social surplus as follows:

\[
S^*(\theta) = \text{Max}\{0, \theta_1 - c\} = \text{Max}\{0, \theta_1 - (\kappa - \theta_2)\}
\]

\[
= \text{Max}\{0, \theta_1 + \theta_2 - \kappa\}, \quad \theta_1 \in \Theta_1, \quad \theta_2 \in \Theta_2.
\]

Note that, by the change of variables, we have reduced the problem to a two agent public goods problem.

Recall that a blocking agent is one for whom it is never efficient to produce if an agent’s value is sufficiently low. Thus if the cost always exceeds the minimum value, the buyer is a blocking agent. Similarly, if buyer’s value is always below the maximum cost, the seller is also a blocking agent. The following result is then a direct implication of Proposition 12.5-3.
Proposition 12.6-1: Trade between blocking agents

If the maximum cost exceeds the maximum value (seller is blocking) and the minimum value is less than the minimum cost (buyer is blocking), then the minimum expected loss of the mechanism designer is equal to the expected gains from trade.

More generally, the following proposition is an immediate implication of Proposition 12.5-2.

Proposition 12.6-2: Myerson-Satterthwaite impossibility Theorem

Suppose that a single buyer’s value is continuously distributed on the interval $\Theta_1 = [\alpha_1, \beta_1]$ and the seller’s cost is continuously distributed on the interval $C = [\gamma, \kappa]$. Suppose also that there are gains from trade for some, but not all realizations. If the cost and value are private and independently distributed, then for efficient exchange between a single buyer and seller the mechanism designer must incur an expected loss.
If there are 2 or more buyers there is a further generalization of this result. Let the set of buyers be
\( \mathcal{I} = \{1, \ldots, I\} \) and let agent \( I+1 \) be the seller. Suppose that the value of each buyer and the cost of the seller all have the same support \([\alpha, \beta]\). Consider the V-C-G mechanism. Maximized total surplus is

\[
S^*(\theta, c) = \max \left\{ \sum_{i=1}^{I} q_i (\theta_i - c) \mid 0 \leq \sum_{i=1}^{I} q_i \leq 1 \right\}
\]

Maximized total surplus when the seller has his worst type \((c = \beta)\) is

\[
S^*(\theta, \beta) = \max \left\{ \sum_{i=1}^{I} q_i (\theta_i - \beta) \mid 0 \leq \sum_{i=1}^{I} q_i \leq 1 \right\}
\]

By hypotheses \( \theta_i \leq \beta \) so \( S^*(\theta, \beta) = 0 \). The seller is therefore a blocking agent.

In the V-C-G mechanism each agent’s equilibrium payoff is the marginal contribution to social surplus. Hence the seller’s equilibrium payoff is

\[
u_{I+1}(\theta, c) = S^*(\theta, c) - S^*(\theta, \beta) = S^*(\theta, c).
\]

Since the seller’s payoff is equal to total surplus the sum of all the payoffs must exceed total surplus so again the V-C-G mechanism generates a loss for the designer.
To complete the proof it must be shown that, for any efficient incentive compatible mechanism, the expected payoff of each buyer and the seller cannot exceed the equilibrium V-C-G payoffs. The proof for each of the buyers is almost identical to that for the agents in the public goods model. A very similar argument holds for the seller as well. Our result is summarized below.

**Proposition 12.6-3: Impossibility theorem with many buyers**

Suppose that each of $n$ buyer’s values and the seller’s cost are all continuously distributed on the interval $\Theta = [\alpha, \beta]$. If the cost and value are private and independently distributed, then for efficient exchange between the buyers and the seller the mechanism designer must incur an expected loss.