A note on the second welfare theorem in an exchange economy

Consider two consumers with the following increasing continuous quasi-concave utility functions.

\[ U^A(x^A) = \sqrt{x^A_1} + \sqrt{x^A_2} \quad \text{and} \quad U^B(x^B) = \begin{cases} -\frac{1}{2}(\beta - x^B_1)^2 + x^B_2, & x^B_1 \leq \beta \\ x^B_2, & x^B_1 > \beta \end{cases} \]

The aggregate endowment is \( \omega \) where \( \omega_A > \beta \).

By the Supporting Hyperplane Theorem there exists a vector \( p > 0 \) such that

\[ U^h(x^h) \geq U^h(\hat{x}^h) \Rightarrow p \cdot x^h \geq p \cdot \hat{x}^h, \quad h \in \{ A, B \}. \]

It is tempting to conjecture that any PE allocation \( E = \{ \hat{x}^A, \hat{x}^B \} \) in such an economy can be decentralized. That is,

\[ U^h(x^h) > U^h(\hat{x}^h) \Rightarrow p \cdot x^h > p \cdot \hat{x}^h, \quad h \in \{ A, B \}. \]

As noted first by Arrow, this is not necessarily true if \( p \cdot \hat{x}^h = 0 \) for some \( h \in \{ A, B \} \). Consider the Edgeworth Box diagram below. An interior PE allocation such as \( \bar{E} \) can be decentralized. That is, any allocation in which one of the consumers is strictly better off is outside that consumer’s budget set.
However, consider the PE allocation $\hat{E}$ on the boundary of the Edgeworth Box as depicted below.

The horizontal green line is now the unique supporting line. That is, for $p = (0,1)$

$$U^h(x^h) \geq U^h(\hat{x}^h) \Rightarrow p \cdot x^h \geq p \cdot \hat{x}^h, \; h \in \{A, B\}.$$ 

Note that $p \cdot \bar{x}^A = 0$. Note also that along the horizontal axis, consumer A’s utility is $U^A(x^A_1, 0) = \sqrt{x^A_1}$.

Thus the requirement for the second welfare theorem,

$$U^h(x^h) > U^h(\hat{x}^h) \Rightarrow p \cdot x^h > p \cdot \hat{x}^h, \; h \in \{A, B\}$$

is not satisfied for consumer $A$.

As we now show, this problem cannot occur if utility is strictly increasing. The key to the proof is the following lemma.
Lemma:

Suppose that for \( \hat{x}^h \) in consumer \( h \)'s consumption set, \( X^h = \mathbb{R}^n_+ \) the vector \( p > 0 \) is supporting at \( \hat{x}^h \) for the upper contour set \( C^U(\hat{x}^h) = \{ x \mid U^h(x^h) \geq U^h(\hat{x}^h) \} \) where \( U^h(\cdot) \) is strictly increasing, that is,

\[
U^h(x^h) \geq U^h(\hat{x}^h) \Rightarrow p \cdot x^h \geq p \cdot \hat{x}^h \tag{1}
\]

Then

\[
p \cdot \hat{x}^h > 0 \ \text{and} \ U^h(x^h) > U^h(\hat{x}^h) \Rightarrow p \cdot x^h > p \cdot \hat{x}^h
\]

Proof: Suppose that for some \( x^h \),

\[
U^h(x^h) > U^h(\hat{x}^h) \ \text{and} \ p \cdot x^h = p \cdot \hat{x}^h. \tag{2}
\]

Since \( p \cdot \hat{x}^h > 0 \), it follows that for \( \lambda \in (0,1) \) \( p \cdot \lambda x^h < p \cdot \hat{x}^h \). Also, since \( U^h(\cdot) \) is continuous it follows that \( U^h(\lambda x^h) > U^h(\hat{x}^h) \) for \( \lambda \) sufficiently close to 1.

But this contradicts (1). Therefore (2) is false. It the follows from (1) that

\[
U^h(x^h) > U^h(\hat{x}^h) \Rightarrow p \cdot x^h > p \cdot \hat{x}^h.
\]

QED

For any aggregate supply \( x \) and allocation \( \{ \hat{x}^h \}_{h=2}^H \) to all consumers other than consumer 1, let \( V^1(x) \) be the maximum utility of consumer 1. That is

\[
V^1(x) = \max \left\{ U^1(x^1) \mid U^h(x^h) \geq U^h(\hat{x}^h), \ h > 1, \ x - \sum_{h=2}^H x^h \geq 0 \right\}
\]

If the utility functions are quasi-concave it follows that that \( V^1(\cdot) \) is quasi-concave as depicted below.\(^1\) Also \( V^1(\cdot) \) is strictly increasing as any increase in \( x \) can be given to consumer 1.

\[^1\] See EM Lemma 3.2-1
The convexity of the upper contour sets of $V^1(\cdot)$ plays the critical role in the proof of the second welfare theorem.

**Second Welfare Theorem for an Exchange Economy**

Consumer $h \in \mathcal{H}$ has a consumption set, $X^h = \mathbb{R}_+^n$. The aggregate endowment $\omega$ is strictly positive. Utility functions $U^h(\cdot)$, $h \in \mathcal{H}$ are continuous, quasi-concave and strictly increasing. Let $\{\hat{x}^h\}_{h \in \mathcal{H}}$ be a PE allocation in which $\hat{x}^h > 0$, $h \in \mathcal{H}$. Then there exists a strictly positive price vector $p$ such that

$$U^h(x^h) > U^h(\hat{x}^h) \Rightarrow p \cdot x^h > p \cdot \hat{x}^h$$

**Proof:** Since $\{\hat{x}^h\}_{h \in \mathcal{H}}$ is a PE allocation, it solves the following optimization problem.

$$V^1(\omega) = \max_{(x^h)_{h \in \mathcal{H}}} \left\{ U^1(x^1) \mid U^h(x^h) \geq U^h(\hat{x}^h), \ h > 1, \ \omega - \sum_{h \in \mathcal{H}} x^h \geq 0 \right\}.$$ 

Moreover, because $U^1(\cdot)$ is strictly increasing, $\hat{x} \equiv \sum_{h \in \mathcal{H}} \hat{x}^h = \omega$.

Because $\omega$ is on the boundary of the set $\{x \mid V^1(x) \geq V^1(\omega)\}$, it follows from the Supporting Hyperplane Theorem that there exists a vector $p \neq 0$, such that

$$V^1(x) \geq V^1(\omega) \Rightarrow p \cdot x \geq p \cdot \omega \quad (3)$$
We now argue that the vector $p$ must be positive. If not, for some $j \ p_j < 0$. Define 
$$
\delta = (0, \ldots, \delta_j, 0, \ldots, 0) > 0 \quad \text{and} \quad x = \hat{x} + \delta.
$$
Since $V^1(\cdot)$ is strictly increasing, $V^1(x) > V^1(\omega)$ and $p \cdot x < p \cdot \omega$. But this contradicts (3), so $p > 0$ after all.

From (3) and the definition of the indirect utility function

$$
U^h(x^h) \geq U^h(\hat{x}^h), \ h \in \mathcal{H} \Rightarrow p \cdot x = p \cdot \sum_{h \in \mathcal{H}} x^h \geq p \cdot \omega.
$$

Since $\hat{x} = \omega$ it follows that $p \cdot \hat{x} = p \cdot \omega$. Therefore

$$
U^h(x^h) \geq U^h(\hat{x}^h), \ h \in \mathcal{H} \Rightarrow p \cdot \sum_{h \in \mathcal{H}} x^h \geq p \cdot \sum_{h \in \mathcal{H}} \hat{x}^h.
$$

Setting $x^k = \hat{x}^k, \ k \neq h$, we may conclude that for consumer $h$,

$$
U^h(x^h) \geq U^h(\hat{x}^h) \Rightarrow p \cdot x^h \geq p \cdot \hat{x}^h. \quad (4)
$$

We now show that $p >> 0$. Since $\hat{x} = \omega >> 0$ and $p > 0$ it follows that

$$
p \cdot \hat{x} = p \cdot \sum_{h \in \mathcal{H}} \hat{x}_h = p \cdot \omega > 0.
$$

Then for some $h$, $p \cdot \hat{x}^h > 0$. Appealing to the Lemma, for this consumer

$$
U^h(x^h) > U^h(\hat{x}^h) \Rightarrow p \cdot x^h > p \cdot \hat{x}^h.
$$

If $p_j = 0$, define $\delta = (0, \ldots, \delta_j, 0, \ldots, 0) > 0$. Since $U^h(\cdot)$ is increasing,

$$
U^h(\hat{x}^h + \delta) > U^h(\hat{x}^h) \quad \text{and} \quad p \cdot (\hat{x}^h + \delta) = p \cdot \hat{x}^h.
$$

But this contradicts (4). Thus $p$ must be strictly positive. Since $\hat{x}^h > 0$ it follows that for all $h \in \mathcal{H}$, $p \cdot \hat{x}^h > 0$. Again appealing to the Lemma

$$
U^h(x^h) > U^h(\hat{x}^h) \Rightarrow p \cdot x^h > p \cdot \hat{x}^h.
$$

QED