5.1 THE ROBINSON CRUSOE ECONOMY

Key ideas: Walrasian equilibrium allocation, optimal allocation, invisible hand at work

A simple economy with production

Two commodities, H consumers, one firm, an aggregate endowment vector $\omega = (\omega_1, 0)$.

Commodity 1 can be consumed or used as an input in the production of commodity 2.

The firm, with strictly convex production set $\mathcal{Y}$ utilizes commodity 1 to produce commodity 2.

Consumers have identical convex homothetic utility function $U^h = U(x_1^h, x_2^h),\ h = 1,...,H$.

Given the assumption of identical homothetic preferences the consumers can be represented by a single consumer "Robinson Crusoe" who has an endowment of ω .

The Optimum

If Robinson Crusoe, with endowment vector ω, chooses the production plan y, his consumption vector is $x = y + \omega$.

The set of feasible consumption bundles is therefore the set $\Upsilon + \omega$. This set is also depicted in Figure 1.

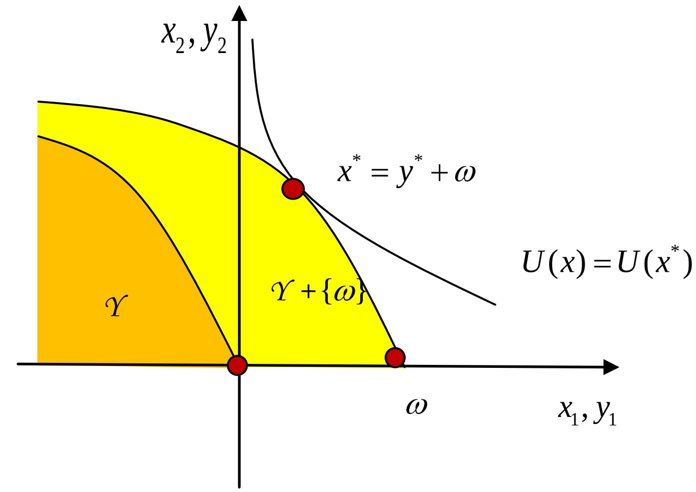

Figure 1: The Optimum

Robinson Crusoe then chooses the consumption bundle x^* that maximizes his utility from the bundles in the set $\Upsilon + \{\omega\}$. Formally, x^* solves the following maximization problem:

$$\underset{x}{Max}\{U(x) \mid x \in \Upsilon + \{\omega\}\}.$$

Example:

$\mathcal{Y} = \{(y_1, y_2) \mid y_1 \le 0,\ \ y_2{}^2 + 64y_1 \le 0\},\ \ U(x) = \ln x_1 + \ln x_2,\ \ \omega = (12,0)$.

We substitute for $x = y + \omega$ and write utility as $U(y+\omega) = \ln(\omega_1 + y_1) + \ln y_2$. Because utility is increasing, the optimum must be on the boundary of the production set so that $y_1 = -y_2{}^2 / 64$.

Substituting for ω and y_1, $U = \ln(12 - y_2{}^2/64) + \ln y_2$.

FOC:

$$\frac{dU}{dy_2} = \frac{-y_2/32}{12 - y_2{}^2/64} + \frac{1}{y_2} = 0. \text{ Hence } \frac{y_2^2}{32} = 12 - \frac{y_2^2}{64}$$

Solving, $y_2^* = 16$ and so $y_1^* = -(y_2^*)^2/64 = -4$

Hence $y^* = (-4,16)$ and $x^* = y^* + \omega = (8,16)$

Supporting hyperplane

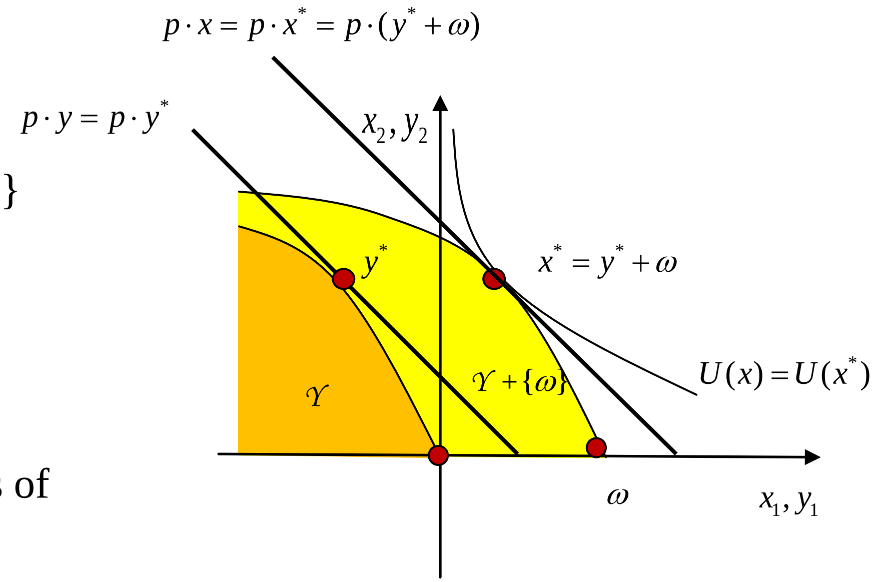

Figure 2: Supporting lines

Since x^* is the optimal consumption bundle the

Interior of the upper contour set $X^* = \{x \mid U(x) \geq U(x^*)\}$

and the set $\Upsilon + \{\omega\}$ is empty. Therefore $x^* = y^* + \omega$

is on the boundary of the sets $\Upsilon + \{\omega\}$ and X^*.

The line $p \cdot x = p \cdot x^*$ is drawn tangent to the boundaries of these two sets.

Since the sets are both convex, the line is a supporting line for both sets. That is

(i) $p \cdot x \leq p \cdot x^*$ for all $x \in \Upsilon + \{\omega\}$ and (ii) $p \cdot x > p \cdot x^*$ for all $x \in \text{int}\, X^*$.

Robinson the Manager

(i) $p \cdot x \leq p \cdot x^*$ for all $x \in \Upsilon + \{\omega\}$

Therefore $p \cdot (y + \omega) \leq p \cdot (y^* + \omega)$ for all $y \in \Upsilon$,

equivalently,

$p \cdot y \leq p \cdot y^*$ for all $y \in \Upsilon$,

Note that with a price vector (p_1, p_2) the total revenue of the

firm is $p_2 y_2$ and total cost is $p_1(-y_1)$ and so the profit of the firm is $\pi(p, y) = p \cdot y$.

Therefore Robinson the manager maximizes the profit of the firm by choosing the production plan y^*.

Crusoe the Consumer

At home Robinson becomes Crusoe the representative consumer. Given the price vector *p*, the value of Crusoe's endowment is $p\cdot\omega$. As the single consumer, Crusoe also receives a dividend equal to the firm's profit. Thus his total income is $p\cdot\omega + p\cdot y^* = p\cdot(\omega + y^*) = p\cdot x^*$ and so his budget constraint is

$$p\cdot x \le p\cdot x^*$$

We have argued that

(ii) $p\cdot x > p\cdot x^*$ for all $x \in \text{int}\, X^*$.

Thus any strictly preferred consumption bundle costs strictly more. Therefore x^* is a maximizer for

$$\underset{x}{Max}\{U(x) \mid p\cdot x \le p\cdot x^* = p\cdot(y^* + \omega)\}$$

The optimal allocation is therefore a WE allocation.

Example (continued):

Robinson the price taking manager

Robinson the manager solves the following maximization problem:

$$\underset{y}{Max}\{p\cdot y \mid y\in\mathcal{Y}\},$$

that is

$$\underset{y}{Max}\{p\cdot y \mid y_1\le 0,\ \ 64y_1+{y_2}^2\le 0\}.$$

For a maximum the constraint must be binding. Then substituting for $y_1=-{y_2}^2/64$, profit is

$$-p_1{y_2}^2/64+p_2y_2.$$

Solving for the profit-maximizing output we obtain,

$y_2(p)=32\dfrac{p_2}{p_1}$. Thus $y_1(p)=-y_2(p)^2/64=-16(\dfrac{p_2}{p_1})^2$ and maximized profit is $\Pi(p)=\dfrac{16{p_2}^2}{p_1}$.

Crusoe the Price-Taking Consumer

Next consider the choice of Crusoe the consumer. The value of his endowment is $p\cdot\omega$. In addition, as the single shareholder in the economy he collects all the dividends $\Pi(p)=p\cdot y(p)$. His spending on corn is therefore constrained as follows: $p\cdot x\le p\cdot\omega+\Pi(p)$.

He therefore solves the following maximization problem.

$$\underset{x}{Max}\{\ln x_1+\ln x_2 \mid p\cdot x\le \Pi(p)+p\cdot\omega\}.$$

Because utility is strictly increasing, the budget constraint must be satisfied with equality at the maximum. From the FOC, and appealing to the Ratio Rule,

$$\frac{\frac{\partial U}{\partial x_1}}{p_1}=\frac{\frac{\partial U}{\partial x_2}}{p_2}\Rightarrow\frac{1}{p_1x_1}=\frac{1}{p_2x_2}=\frac{2}{p_1x_1+p_2x_2}=\frac{2}{p\cdot\omega+\Pi(p)}.$$

Therefore $x_2(p)=\frac{1}{2}(\frac{\Pi(p)+p\cdot\omega}{p_2})$.

Walrasian Equilibrium

We have already seen that $y_2^*(p) = 32\frac{p_2}{p_1}$ and $\Pi(p) = \frac{16{p_2}^2}{p_1}$.

Therefore demand for commodity 2 is $x_2(p) = \frac{1}{2}(\frac{16p_2}{p_1} + \frac{12p_1}{p_2})$.

It follows that excess demand for commodity 2 is

$$z_2(p) = x_2(p) - y_2(p) = (\frac{8p_2}{p_1} + \frac{6p_1}{p_2}) - \frac{32p_2}{p_1} = 6\frac{p_1}{p_2}(1-4(\frac{p_2}{p_1})^2$$

$$= 0 \text{ if } \frac{p_2}{p_1} = \frac{1}{2}.$$

Thus $\bar{p} = (2,1)$ is a WE price vector and the WE allocation is $x^*(\bar{p}) = (16,8)$.

Class Exercise: Why must the other market clear as well?

Exercise: Show that the WE allocation is the same if there are 4 identical firms, each with a production set $\Upsilon^f = \{y^f \mid 16{y_1}^f + (y_2^f)^2 \le 0\}$.