5.1 THE ROBINSON CRUSOE ECONOMY

Key ideas: Walrasian equilibrium allocation, optimal allocation, invisible hand at work

A simple economy with production

Two commodities, $H$ consumers, one firm, an aggregate endowment vector $\omega = (\omega_1, 0)$.

Commodity 1 can be consumed or used as an input in the production of commodity 2.

The firm, with strictly convex production set $\mathcal{Y}$ utilizes commodity 1 to produce commodity 2.

Consumers have identical convex homothetic utility function $U^h = U(x_1^h, x_2^h)$, $h = 1, \ldots, H$.

Given the assumption of identical homothetic preferences the consumers can be represented by a single consumer “Robinson Crusoe” who has an endowment of $\omega$. 
The Optimum

If Robinson Crusoe, with endowment vector $\omega$, chooses the production plan $y$, his consumption vector is $x = y + \omega$.

The set of feasible consumption bundles is therefore the set $\mathcal{Y} + \omega$. This set is also depicted in Figure 1.

Robinson Crusoe then chooses the consumption bundle $x^*$ that maximizes his utility from the bundles in the set $\mathcal{Y} + \{\omega\}$. Formally, $x^*$ solves the following maximization problem:

$$\max_x \{ U(x) \mid x \in \mathcal{Y} + \{\omega\} \}.$$
Example:
\[ Y = \{(y_1, y_2) \mid y_1 \leq 0, \ y_2^2 + 64y_1 \leq 0\}, \quad U(x) = \ln x_1 + \ln x_2, \quad \omega = (12, 0). \]

We substitute for \( x = y + \omega \) and write utility as \( U(y + \omega) = \ln(\omega_1 + y_1) + \ln y_2 \). Because utility is increasing, the optimum must be on the boundary of the production set so that \( y_1 = -y_2^2 / 64 \).

Substituting for \( \omega \) and \( y_1 \), \( U = \ln(12 - y_2^2 / 64) + \ln y_2 \).

FOC:
\[
\frac{dU}{dy_2} = \frac{-y_2 / 32}{12 - y_2^2 / 64} + \frac{1}{y_2} = 0. \quad \text{Hence} \quad \frac{y_2^2}{32} = 12 - \frac{y_2^2}{64}
\]

Solving, \( y_2^* = 16 \) and so \( y_1^* = -(y_2^*)^2 / 64 = -4 \)

Hence \( y^* = (-4, 16) \) and \( x^* = y^* + \omega = (8, 16) \).
Supporting hyperplane

Since \( x^* \) is the optimal consumption bundle the
Interior of the upper contour set \( X^* = \{ x \mid U(x) \geq U(x^*) \} \)
and the set \( Y + \{ \omega \} \) is empty. Therefore \( x^* = y^* + \omega \)
is on the boundary of the sets \( Y + \{ \omega \} \) and \( X^* \).
The line \( p \cdot x = p \cdot x^* \) is drawn tangent to the boundaries of
these two sets.
Since the sets are both convex, the line is a supporting line
for both sets. That is

(i) \( p \cdot x \leq p \cdot x^* \) for all \( x \in Y + \{ \omega \} \) and
(ii) \( p \cdot x > p \cdot x^* \) for all \( x \in \text{int } X^* \).
Robinson the Manager

(i) \( p \cdot x \leq p \cdot x^* \) for all \( x \in \mathcal{Y} + \{\omega\} \)

Therefore \( p \cdot (y + \omega) \leq p \cdot (y^* + \omega) \) for all \( y \in \mathcal{Y} \),
equivalently,
\( p \cdot y \leq p \cdot y^* \) for all \( y \in \mathcal{Y} \),

Note that with a price vector \((p_1, p_2)\) the total revenue of the
firm is \( p_2y_2 \) and total cost is \( p_1(-y_1) \) and so the profit of the firm is \( \pi(p, y) = p \cdot y \).

Therefore Robinson the manager maximizes the profit of the firm by choosing the production plan \( y^* \).
**Crusoe the Consumer**

At home Robinson becomes Crusoe the representative consumer. Given the price vector \( p \), the value of Crusoe’s endowment is \( p \cdot \omega \). As the single consumer, Crusoe also receives a dividend equal to the firm’s profit. Thus his total income is \( p \cdot \omega + p \cdot y^* = p \cdot (\omega + y^*) = p \cdot x^* \) and so his budget constraint is

\[
p \cdot x \leq p \cdot x^*
\]

We have argued that

(ii) \( p \cdot x > p \cdot x^* \) for all \( x \in \text{int } X^* \).

Thus any strictly preferred consumption bundle costs strictly more. Therefore \( x^* \) is a maximizer for

\[
\max_{x} \{U(x) \mid p \cdot x \leq p \cdot x^* = p \cdot (y^* + \omega)\}
\]

The optimal allocation is therefore a WE allocation.
Example (continued):

Robinson the price taking manager

Robinson the manager solves the following maximization problem:

\[ \max_y \{ p \cdot y \mid y \in Y \} , \]

that is

\[ \max_y \{ p \cdot y \mid y_1 \leq 0, 64y_1 + y_2^2 \leq 0 \} . \]

For a maximum the constraint must be binding. Then substituting for \( y_1 = -y_2^2 / 64 \), profit is

\[ -p_1y_2^2 / 64 + p_2y_2 . \]

Solving for the profit-maximizing output we obtain,

\[ y_2(p) = 32 \frac{p_2}{p_1} . \] Thus \( y_1(p) = -y_2(p)^2 / 64 = -16 \left( \frac{p_2}{p_1} \right)^2 \) and maximized profit is \( \Pi(p) = \frac{16p_2^2}{p_1} . \)
**Crusoe the Price-Taking Consumer**

Next consider the choice of Crusoe the consumer. The value of his endowment is \( p \cdot \omega \). In addition, as the single shareholder in the economy he collects all the dividends \( \Pi(p) = p \cdot y(p) \). His spending on corn is therefore constrained as follows: \( p \cdot x \leq p \cdot \omega + \Pi(p) \).

He therefore solves the following maximization problem.

\[
\text{Max}_{x} \left\{ \ln x_1 + \ln x_2 \mid p \cdot x \leq \Pi(p) + p \cdot \omega \right\}.
\]

Because utility is strictly increasing, the budget constraint must be satisfied with equality at the maximum. From the FOC, and appealing to the Ratio Rule,

\[
\frac{\partial U}{\partial x_1} = \frac{\partial U}{\partial x_2} \Rightarrow \frac{1}{p_1} = \frac{1}{p_2} = \frac{2}{p_1x_1 + p_2x_2} = \frac{2}{p \cdot \omega + \Pi(p)}.
\]

Therefore \( x_2(p) = \frac{1}{2} \left( \frac{\Pi(p) + p \cdot \omega}{p_2} \right) \).
Walrasian Equilibrium

We have already seen that \( y_2^*(p) = 32 \frac{p_2}{p_1} \) and \( \Pi(p) = \frac{16 p_2^2}{p_1} \).

Therefore demand for commodity 2 is \( x_2(p) = \frac{1}{2} \left( \frac{16 p_2}{p_1} + \frac{12 p_1}{p_2} \right) \).

It follows that excess demand for commodity 2 is

\[
z_2(p) = x_2(p) - y_2(p) = \left( \frac{8 p_2}{p_1} + \frac{6 p_1}{p_2} \right) - \frac{32 p_2}{p_1} = 6 \frac{p_1}{p_2} \left( 1 - 4 \left( \frac{p_2}{p_1} \right)^2 \right)
\]

\[
= 0 \quad \text{if} \quad \frac{p_2}{p_1} = \frac{1}{2}.
\]

Thus \( \bar{p} = (2,1) \) is a WE price vector and the WE allocation is \( x^*(\bar{p}) = (16,8) \).

Class Exercise: Why must the other market clear as well?

Exercise: Show that the WE allocation is the same if there are 4 identical firms, each with a production set \( \mathcal{Y}^f = \{ y^f | 16y_1^f + (y_2^f)^2 \leq 0 \} \).