5.2: EQUILIBRIUM AND EFFICIENCY WITH PRODUCTION

Key ideas: Walrasian equilibrium, first and second welfare theorems

A general model
First welfare Theorem
Second welfare theorem
A general model

**Firms**

Firm $f$, $f = 1,\ldots,F$ has a set of feasible production plans $\mathcal{Y}^f \subset \mathbb{R}^n$ and chooses a production vector $y^f = (y_1^f,\ldots,y_n^f) \in \mathcal{Y}^f$. A production plan for the economy $\{y^f\}_{f=1}^F$ is a plan for each of the firms.

The aggregate production plan for the economy is the sum of all the individual plans $y = \sum_{f=1}^F y^f$.

The set of all feasible aggregate production plans, $\mathcal{Y}$, is the aggregate production set.
Consumers

Commodities are private, that is, consumer $h$ has preferences over his own consumption vector $x^h = (x^h_1, \ldots, x^h_n)$ and not those of other consumers. Let $X^h \subseteq \mathbb{R}^n$ be the consumption set of consumer $h$, $h = 1, \ldots, H$. That is, preferences are defined over $X^h$. We assume that consumer $h$ has an endowment vector $\omega^h \in X^h$. A consumption allocation in this economy $\{x^h\}_{h=1}^H$ is an allocation of consumption bundles $x^h \in X^h$, $h = 1, \ldots, H$. The aggregate consumption in the economy is the sum of the individual consumption vectors $x = \sum_{h=1}^H x^h$. Similarly the aggregate endowment is $\omega = \sum_{h=1}^H \omega^h$.

Shareholdings

Firms are owned by consumers. Consumer $h$ has an ownership share in firm $f$ of $\theta_{hf}$. Ownership shares must sum to 1, that is

$$\sum_{h=1}^H \theta_{hf} = 1, \quad f = 1, \ldots, F.$$
**Feasible allocation**

Given the aggregate demand $x$, endowment $\omega$ and supply $y$, define excess demand $z = x - \omega - y$. An allocation is feasible if aggregate excess demand is negative

$$z = x - \omega - y \leq 0.$$

**Pareto efficient allocation**

A feasible plan for the economy $\{\hat{x}^h_{h=1}^{H}, \hat{y}^f_{f=1}^{F}\}$ is Pareto efficient (PE) if there is no other feasible plan $\{x^h_{h=1}^{H}, y^f_{f=1}^{F}\}$ that is strictly preferred by at least one consumer and weakly preferred by all consumers.
Price taking

Let $p > 0$ be the price vector. Consumers and firms are price takers. Thus if firm $f$ chooses the production plan $y^f$ it has a profit of $p \cdot y^f$. Consumer $h$ receives her share of the profit as a dividend payment. The total dividend payment received by consumer $h$ is therefore $\sum_f \theta^{hf} p \cdot y^f$. Adding the value of her endowment, consumer $h$ has a wealth $W^h = p \cdot \omega^h + \sum_f \theta^{hf} p \cdot y^f$.

She then chooses a consumption bundle $\underline{x}^h$ in her budget set $\{x^h \in X^h \mid p \cdot x^h \leq W^h\}$.

Note that her budget set is largest if wealth is maximized. Thus, as a shareholder, consumer $h$’s interests are best served by firm managers who maximize profit.
**Walrasian equilibrium**

Given the price vector $p$, let $\overline{y}^f$, $f = 1, \ldots, F$, be a production plan that maximizes the profit of firm $f$. That is,

$$p \cdot \overline{y}^f \geq p \cdot y^f, \text{ for all } y^f \in Y^f, \ f = 1, \ldots, F. \quad (*)$$

Also, let $\overline{x}^h$ be a most preferred consumption plan in consumer $h$’s budget set. That is,

$$U^h(\overline{x}^h) \geq U^h(x^h), \text{ for all } x^h \text{ such that } p \cdot x^h \leq W^h \quad (**)$$

The aggregate excess demand vector is then $\overline{z} = \overline{x} - \omega - \overline{y}$.

**Definition: Walrasian equilibrium prices**

The price vector $p \geq 0$ is a WE price vector if for some $\{\overline{y}^f\}_{f=1}^F$ satisfying $(*)$ and $\{\overline{x}^h\}_{h=1}^H$ satisfying $(**)$, the excess demand vector is negative ($\overline{z} \leq 0$). Moreover $p_j = 0$ for any market in which excess demand is strictly negative ($\overline{z}_j < 0$).
Proposition 5.2-1: First welfare theorem

If the preferences of each consumer satisfy the local non-satiation postulate, the Walrasian equilibrium allocation is Pareto efficient.

Proof: We appeal to the Duality Lemma (see EM 2.3).

Lemma 2.2-3: Duality Lemma

If the local non-satiation assumption holds and \( x^* \in \arg \max \{ U(x) \mid x \geq 0, \ p \cdot x \leq I \} \), then

\[
U(x) \geq U(x^* ) \implies p \cdot x \geq p \cdot x^* \text{ and so } x^* \in \arg \min \{ p \cdot x \mid x \geq 0, \ U(x) \geq U(x^*) \}. \tag{5.2-1}
\]

In addition, note that since \( \bar{x}^h \) maximizes utility over consumer \( h \)'s budget set, any strictly preferred bundle \( x^h \) must cost strictly more than the equilibrium allocation. That is,

\[
U^h(x^h) > U^h(\bar{x}^h) \implies p \cdot x^h > p \cdot \omega^h + p \sum_{f=1}^{F} \theta^{hf} \bar{y}^f \neq . \tag{5.2-2}
\]

Consider any allocation \( \{ x^h \}_{h=1}^{H} \) that is Pareto preferred to the WE allocation \( \{ \bar{x}^h \}_{h=1}^{H} \). Summing over consumers, it follows that

\[
\sum_{h=1}^{H} p \cdot x^h > \sum_{h=1}^{H} (p \cdot \omega^h + p \sum_{f=1}^{F} \theta^{hf} \bar{y}^f ) .
\]
Since shares sum to 1, this can be rewritten as follows.
\[ p \cdot x > p \cdot \omega + p \cdot \bar{y}. \]

Therefore
\[ p \cdot (x - \omega - \bar{y}) > 0, \]

Also \(\bar{y}^f\) is profit maximizing over \(Y^f\). Hence \(p \cdot \bar{y}^f \geq p \cdot y^f\). Therefore
\[ p \cdot (x - \omega - y) \geq p \cdot (x - \omega - \bar{y}) > 0. \]

(5.2-3)

Suppose that the allocation \(\{x_h^H\}_{h=1}^{H}, \{y_f^F\}_{f=1}^{F}\) is feasible. We will show, by contradiction, that no such allocation exists.

For feasibility, excess demands must be negative so \(z = x - \omega - y \leq 0\).

Then, since the Walrasian equilibrium price vector is positive,
\[ p \cdot (x - \omega - y) \leq 0. \]

But this contradicts (5.2-3). Thus there is no Pareto preferred feasible allocation.

Q.E.D.
**Decentralization Theorem**

Next we consider the second welfare theorem with production. We follow the same line of argument as in the proof for the exchange economy. However we no longer assume that consumption bundles are necessarily positive. Thus consumers may supply commodities (e.g. labor services) and production vectors have both positive components (outputs) and negative components (inputs.)

**Proposition 5.2-2: Second welfare theorem with production**

Let \( \{ \hat{x}^h \}_{h=1}^H, \{ \hat{y}^f \}_{f=1}^F \) be a Pareto efficient allocation. Suppose

(a) consumption vectors are private,
(b) consumption sets \( X^h, \ h = 1, \ldots, H \) are convex
(c) utility functions are continuous, quasi-concave and satisfy the local non-satiation property
(d) for each \( h \) there is some \( \underline{x}^h \in X^h \) such that \( \underline{x}^h < \hat{x}^h \), and
(e) production sets \( Y^f, \ f = 1, \ldots, F \) are convex and satisfy the free disposal property.

Then there exists a price vector \( p > 0 \) such that

\[
\begin{align*}
\hat{x}^h &> \underline{x}^h \Rightarrow p \cdot \hat{x}^h > p \cdot \underline{x}^h, \ h=1,\ldots,H \\
y^f &\in Y^f \Rightarrow p \cdot y^f \leq p \cdot \hat{y}^f.
\end{align*}
\]
Proof: Let \( \{\hat{x}^h\}_{h=1}^H, \{\hat{y}^f\}_{f=1}^F \) be a PE allocation. As in the pure exchange economy we introduce the indirect utility function

\[
V^1(x) = \text{Max} \{U^1(x^1) \mid \sum_{h=1}^H x^h \leq x, \quad U^h(x^h) \geq U^h(\hat{x}^h), \quad h = 2,...,H \}.
\]

As argued for the exchange economy (see Chapter 3), \( V(x) \) is quasi-concave. Also since \( U^1 \) has the local non-satiation property, so does \( V \).

We also define the set of feasible aggregate consumption vectors \( Z = \{\omega\} + \mathcal{Y} \). This is depicted for a two commodity example in which some of the initial endowment of commodity 1 is transformed into commodity 2.

Note that since production sets are convex, so is the aggregate production set.
Consider the following optimization problem:

\[
\text{Max } \{ V^1(x) \mid x \in Z \}. \quad (5.2-4)
\]

Since \( \{ \hat{x}^h \}_{h=1}^H, \{ \hat{y}^f \}_{f=1}^F \) is Pareto efficient, \( U^1(\hat{x}^1) \) is the highest feasible utility for consumer 1, given that no other consumer is made worse off than in the PE allocation.

Thus \( \{ \hat{x}^h \}_{h=1}^H, \{ \hat{y}^f \}_{f=1}^F \) solves this maximization problem and so \( V^1(\hat{x}) = U^1(\hat{x}^1) \).

The solution is depicted in the figure. Given local non-satiation, the maximizing aggregate consumption vector \( \hat{x} = \omega + \hat{y} \) must lie on the boundary of \( Z \).

Figure 1: Supporting hyperplane
Step 1: Appeal to the Supporting Hyperplane Theorem

Define $\hat{X} \equiv \{x \mid V^1(x) \geq V^1(\hat{x})\}$. Then $\text{int} \hat{X} \cap Z = \emptyset$ so that $\hat{x} = \omega + \hat{y}$ is also on the boundary of $\hat{X}$. Since both $Z$ and $\hat{X}$ are convex it follows that $\hat{X} - Z$ is convex. Moreover $\hat{x} - \omega - \hat{y} = 0$ lies on the boundary of $\hat{X} - Z$.

It follows from the Supporting Hyperplane Theorem\(^1\) that there exists a supporting vector $p \neq 0$ such that

$$p \cdot (x - \omega - y) \geq p \cdot (\hat{x} - \omega - \hat{y}) = 0 \text{ for all } x \in \hat{X} \text{ and } y \in Y$$

Setting $x = \hat{x}$,

(i) $p \cdot y \leq p \cdot \hat{y}$ for all $y \in Y$.

Setting $y = \hat{y}$,

(ii) $p \cdot x \geq p \cdot \hat{x}$ for all $x \in \hat{X}$.

\(^1\) See EM Proposition 1.1-1 and Appendix B.
Step 2: Establish that the vector $p$ must be positive

We suppose that some components of $p$ are negative and show that this yields to a contradiction of (i). Define $\delta = (\delta_1, \ldots, \delta_n) > 0$ such that $\delta_j > 0$ if and only if $p_j < 0$. Consider the vector $y = \hat{y} - \delta$. Given free disposal $\hat{y} - \delta \in Y$.

Also $p \cdot y = p \cdot \hat{y} - p \cdot \delta > p \cdot \hat{y}$ since $p \cdot \delta < 0$.

But this violates (i). Thus $p > 0$ after all.
Step 3: Show that \( \hat{y}^f \) is profit-maximizing.

Next note that we can rewrite conditions (i) and (ii) as follows:

\[
(i) \quad y \in \mathcal{Y} \Rightarrow \sum_{f=1}^{F} p \cdot y^f \leq \sum_{f=1}^{F} p \cdot \hat{y}^f \quad \text{and} \quad (ii) \quad V(x) \geq V(x) \Rightarrow \sum_{h=1}^{H} p \cdot x^h \geq \sum_{h=1}^{H} p \cdot \hat{x}^h.
\]

Setting \( y^j = \hat{y}^j, \quad j \neq f \) in (i) it follows that

\[
(i') \quad y^f \in \mathcal{Y} \Rightarrow p \cdot y^f \leq p \cdot \hat{y}^f, \quad f = 1, \ldots, F.
\]

Thus \( \hat{y}^f \) is profit maximizing for firm \( f \).
Step 4: Show that $\hat{x}^h$ is utility-maximizing.

Similarly, setting $x^i = \hat{x}^i$, $i \neq h$ in (ii) on the previous slide, it follows that\(^2\)

(ii)' \quad U^h(x^h) \geq U^h(\hat{x}^h) \Rightarrow p \cdot x^h \geq p \cdot \hat{x}^h.

The final step is to show that if $U^h(x^h) > U^h(\hat{x}^h)$ then $p \cdot x^h > p \cdot \hat{x}^h$. Suppose instead that

$U^h(x^h) > U^h(\hat{x}^h)$ and $p \cdot x^h = p \cdot \hat{x}^h$.

Consider the consumption vector $\tilde{x}^h = (1 - \lambda)x^h + \lambda \hat{x}^h$, where $\lambda \in (0, 1)$. Since $\hat{x}^h \in X^h$ and $X^h$ is convex, $\tilde{x}^h \in X^h$. Since $x^h < \hat{x}^h$,

$p \cdot \tilde{x}^h = (1 - \lambda)p \cdot x^h + \lambda p \cdot \hat{x}^h < (1 - \lambda)p \cdot x^h + \lambda p \cdot \hat{x}^h = p \cdot \hat{x}^h.$

Also, given the continuity of preferences, $U^h(\tilde{x}^h) > U^h(\hat{x}^h)$ for $\lambda$ sufficiently close to 1. But this is impossible since condition (ii)' is violated. \(\text{QED}\)

\(^2\)From the definition of the indirect utility function $U^h(x^h) \geq U^h(\hat{x}^h)$, $h = 2, \ldots, H$.

Also $U^1(x^i) = V^1(x) \geq V^1(\hat{x}) = U^1(\hat{x}^i)$. 