EQUILIBRIUM IN FINANCIAL MARKETS

8.1 ARROW-DEBREU EQUILIBRIUM

Key ideas: trading in state-contingent markets to spread risk, security markets as a substitute for state-contingent markets

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**Introductory example**

Alex and Bev live on a volcanic island. Alex owns a coconut plantation to the east and Bev owns one to the west of a volcano. The volcano will erupt and one of the plantations will be damaged.

In state 1 the ash will damage Alex’s plantation and in state 2 Bev’s plantation. Thus each consumer has an endowment of coconuts that is state dependent.

Let $\omega_s^h \in \{A, B\}$ be the state $s$ endowment of agent $h$. Initially we assume that the damage is the same in the two states. Therefore $(\omega_1^A, \omega_2^A) = (\alpha - L, \alpha)$ and $(\omega_1^B, \omega_2^B) = (\beta, \beta - L)$ . Thus the aggregate endowment is $\omega = (\alpha + \beta - L, \alpha + \beta - L)$ .

Let the probability of state $s$ be $\pi_s$. Then, if consumer $h$ has a von Neumann Morgenstern (VNM) utility function $v^h(\cdot)$ and is allocated a final consumption bundle $x^h = (x_1^h, x_2^h)$, the consumer’s expected utility is

$$U^h(x^h) = \sum_{s=1}^{2} \pi_s v^h(x_s^h), \ h \in \{A, B\}.$$
Pareto Efficient Allocations

Consumer $h$ thus has a marginal rate of substitution (MRS)

$$MRS^h(x_1, x_2) = \frac{\partial U^h}{\partial x_1^h} / \frac{\partial U^h}{\partial x_2^h} = \frac{\pi_1 v'_h(x_1^h)}{\pi_2 v'_h(x_2^h)} = \frac{\pi_1}{\pi_2}$$

along the $45^\circ$ line.

With no aggregate uncertainty, the Edgeworth Box is as depicted.

Note that the diagonal is the $45^\circ$ line for each consumer. Thus, the indifference curves are tangential along this line. It follows that the Pareto-efficient (PE) allocations must lie along the diagonal. As long as each consumer is holding risk, both gain from shedding some of that risk. Setting this in a market context, Alex would like to trade away some of his state 2 endowment for additional claims in state 1. In contrast, Bev would like to trade some of her state 1 endowment for additional claims in state 2. Each seeks to insure against the possible loss.
Suppose that insurance companies offer units of coverage in state 1 at a price per unit of \( r' \) in state 2. A one unit trade changes the insurance companies endowment by \((-1, r')\).

Similarly, if the insurance companies offer units of coverage in state 2 at a price per unit of \( r'' \) in state 1, a one unit trade changes the insurance companies endowment by \((r'',-1)\) and so a trade of \(1/r''\) units of coverage changes the insurance companies endowment by \((1,-1/r'')\). Summing these endowment changes, the total change in endowment is \((0,r'-1/r'')\).

Thus there is a sure gain unless \( r'=1/r'' \). Then, in Walrasian Equilibrium insurance markets there is a unique implicit price ratio \( \frac{p_1}{p_2} = r' = \frac{1}{r''} \).

Buying insurance against the possibility of loss is therefore equivalent to trading in state claims markets at prices \( p = (p_1, p_2) \). If individual \( h \) trades these claims, she seeks to solve the following problem:

\[
\text{Max}_{\pi} \{ U^h(x^h) = \sum_{s=1}^{2} \pi_s v_h(x^h_s) \mid p \cdot x^h \leq p \cdot \omega^h \}.
\]
Except for the interpretation, this problem is completely standard. By the First welfare theorem, the WE must be a PE allocation. But we have just argued that for efficiency \( c_1^h = c_2^h \). Thus the equilibrium price ratio is

\[
\frac{p_1}{p_2} = \frac{\pi_1 v_h'(c_1^h)}{\pi_2 v_h'(c_2^h)} = \frac{\pi_1}{\pi_2}.
\]

That is, the equilibrium price ratio is equal to the probability ratio or “odds”.

Suppose next that the loss is bigger in state 2.

Then the aggregate endowment is larger in state 1.

This is depicted in Figure 8.1-2.

Below Alex’s certainty line, \( MRS^A(x^A) < \frac{\pi_1}{\pi_2} \), and below Bev’s certainty line \( MRS^B(x^B) > \frac{\pi_1}{\pi_2} \).

Thus, no efficient allocation lies below both certainty lines. An identical argument establishes that no efficient allocation lies above both certainty lines. Thus the efficient allocations lie in the shaded region depicted in Figure 8.1-2, between the certainty lines.
There are two immediate implications. First, at any efficient allocation, both Alex and Bev are allocated more state 1 claims than state 2 claims. Thus, the aggregate risk is shared. Second, it follows that for all efficient allocations,

\[ MRS^h(x^h) = \frac{\pi_1 V_h'(x_1^h)}{\pi_2 V_h'(x_2^h)} < \frac{\pi_1}{\pi_2}. \]

Hence, the market equilibrium price of state claims \( \frac{p_1}{p_2} = MRS^h(x^h) = \frac{\pi_1 V_h'(x_1^h)}{\pi_2 V_h'(x_2^h)} < \frac{\pi_1}{\pi_2}. \) Thus the market prices of state claims reflect the relative shortage of state 2 output.

**Class Exercise**

In an \( S \) state \( H \) consumer economy with identical beliefs suppose that \( \omega_1 < \omega_2 < \ldots < \omega_S \). Show that for any \( s \) and \( t > s \), if \( \frac{p_s}{\pi_s} < \frac{p_t}{\pi_t} \) then aggregate demand for state claims is higher in state \( s \) than in state \( t \).

Hence show that in a WE \( \frac{p_1}{\pi_1} > \frac{p_2}{\pi_2} > \ldots > \frac{p_S}{\pi_S} \).
Arrow-Debreu Equilibrium

We now consider a general economy of $H$ consumers $F$ firms and $S$ states. At each date and in each state there are $L$ commodities. We extend the basic equilibrium model by indexing each commodity by the date of its delivery and the state in which it is delivered. For simplicity we consider a two-period world. Consumer $h$ has a date zero consumption vector of $x^h_0 = (x^h_{01}, ..., x^h_{0L})$. Uncertainty is resolved after date 0 and before date 1. If state $s$ is realized, consumer $h$ has a date 1 consumption vector of $x^h_s = (x^h_{s1}, ..., x^h_{sL})$. Figure 8.1-3 depicts the uncertain consumption vectors of consumer $h$ in tree form.

![Diagram](image)

Figure 8.1-3: State contingent consumption by consumer $h$
**Commodities:**

At date 0 and in each of the \(S\) states at date 1 there are \(L\) commodities thus there are \(L + SL = (S + 1)L\) commodities in this economy. We write the entire consumption vector as

\[ x^h = (x^h_0, x^h_1, \ldots, x^h_S) \in X^h \subseteq \mathbb{R}^{(S+1)L} \].

Similarly we write consumer \(h\)'s endowment vector as

\[ \omega^h = (\omega^h_0, \omega^h_1, \ldots, \omega^h_S) \].

**Firms:**

Let \(y^f_0\) be the period 1 production vector of firm \(f\) and let \(y^f_s\) be the period 2 production vector in state \(s\) of firm \(f\). Thus the production vector of the firm is

\[ y^f = (y^f_0, y^f_1, \ldots, y^f_S) \in \mathbb{R}^{(S+1)L} \]

where each component is itself an \(L\) vector, that is \(y^f \in \mathbb{R}^{(S+1)L}\). Firm \(f\) has a production set \(\mathcal{Y}^f\).
Consumers:

Given the independence axiom plus the basic axioms of consumer choice, there exists a continuous utility function $u^h(x^h_0, x^h_s)$ such that preferences over these risky prospects can be expressed in the following separable expected utility form

$$U^h(x^h) = \sum_{s=1}^{S} \pi^h_s u^h(x^h_0, x^h_s).$$

We assume that utility is strictly increasing in at least one commodity. Finally let $\theta^hf$ be consumer $h$’s shareholding of firm $f$.

**Feasible allocation:** An allocation $\{x^h \in X^h\}_{h=1}^{H}, \{y^f \in Y^f\}_{f=1}^{F}$ is feasible if

$$\sum_{h=1}^{H} x^h \leq \sum_{h=1}^{H} \omega^h + \sum_{f=1}^{F} y^f.$$
A-D Equilibrium

A WE of this economy can then be defined in the usual way. That is we introduce markets for each of the \((S+1)L\) commodities. The allocation \(\{x^h \in X^h\}_{h=1}^H, \{y^f \in Y^f\}_{f=1}^F\) is a WE allocation if for some \(p > 0\),

(i) \[ p \cdot x^h \leq p \cdot \omega^h + \sum_{f=1}^F \theta_{hf} \cdot p \cdot y^f \]

consumption allocations are in budget sets

(ii) \[ p \cdot \hat{y}^f > p \cdot y^f \Rightarrow \hat{y}^f \notin Y^f \]

the profit of each firm is maximized

(ii) \[ U^h(\hat{x}^h) > U^h(x^h) \Rightarrow p \cdot \hat{x}^h > p \cdot x^h \]

no strictly preferred allocation is in a consumer’s budget set

(iii) \[ \sum_{h=1}^H x^h = \sum_{h=1}^H \omega^h + \sum_{f=1}^F y^f \]

markets clear
Welfare theorems

The two welfare theorems then apply immediately. An Arrow-Debreu (A-D) equilibrium allocation is PE, and given convex preferences and production sets, any PE allocation can be achieved as an A-D equilibrium via an appropriate redistribution of income.

Example: A-D Equilibrium with Production

Alex owns a firm with a state-contingent output of (140,80). Bev owns a firm with non-contingent production set $Y = \{(z,q) \mid q \leq 80 - z^2, \; z \geq 0\}$. The two states are equally likely. Each individual has a VNM utility function

$$U^h(x^i_1, x^i_2) = \pi_1 \ln(x^i_1) + \pi_2 \ln(x^i_2).$$

In principle we could solve this problem as follows. First, solve for Bev’s profit maximizing outputs given state claims prices $p = (p_1, p_2)$. This yields the aggregate supply vector $y(p)$. Second, compute Bev’s profit and the value of Alex’s endowment at these prices. We can then solve for the consumption choice of each consumer at these prices and hence solve for aggregate demand $x(p)$. Finally, choose a price ratio that equates supply and demand.
Instead there is a convenient short-cut. We note that Alex and Bev have the same homothetic preferences. Then we can treat the economy as a Robinson Crusoe economy with just one individual. Aggregate supply is \((220 - z^2 / 20, 80 + z), \ z \geq 0\). The expected utility of the representative consumer is therefore 

\[
U^R = \frac{1}{2} \ln(220 - z^2 / 20) + \frac{1}{2} \ln(80 + z).
\]

It is readily checked that this is maximized at \(z^* = 20\) and hence that aggregate supply is \((200, 100)\).
To solve for the equilibrium prices we seek prices such that \( z^* \) is indeed the profit-maximizing plan.

Iso-profit lines and the production possibility frontier of Bev’s firm are depicted in Figure 8.1-4.

Along the frontier we have

\[
(y_1(z), y_2(z)) = (80 - z^2 / 20, z).
\]

Thus, the slope of the frontier at \( z^* \) is

\[
\frac{dy_2}{dy_1} = \frac{y_2'(z^*)}{y_1'(z^*)} = \frac{10}{-z^*} = \frac{1}{2}.
\]

Thus, the slopes of the iso-profit line and the frontier are equal at \( z^* \) if the price ratio \( p_1 / p_2 = 1 / 2 \).

Suppose we set the price of state 1 claims equal to 1. Then \( p_2 = 2 \). The value of Alex’s firm is

\[
P^A = (1,2) \cdot (140,80) = 300,
\]

and the value of Bev’s firm is \( P^B = p \cdot y(z) = 100 \).

Figure 8.1-4: Bev’s optimal production and consumption
Trading securities

Thus far we have focused on trading in state claims markets. Suppose instead that the two individuals trade shares in the two firms. Given her initial shareholding of 100% of the shares in firm 2, Bev’s wealth is 100. If she does no trading, her final consumption is the output of the firm (60,20). Alternatively, she could sell her firm and buy a fraction $100/300 = 1/3$ of firm 1, thus giving her one-third of the total dividend stream from firm 1. The consumption bundles associated with these two non-diversified portfolios are depicted in Figure 8.1-5.
A third alternative is to purchase one-quarter of the shares of firm 1 at a cost of 75. This leaves Bev holding shares worth 25 in firm 2; that is, a 25% holding. Her final dividend is then one-quarter of the total output in each state; that is, (50,25). Thus, by trading in the asset markets, Bev is able to achieve the same outcome as she would have by trading in state-claims markets.

As this example suggests, in certain circumstances trading in security markets achieves the same allocation achieved in an Arrow-Debreu equilibrium in which consumers trade directly in contingent claims markets.