8.3 CAPITAL ASSET PRICING MODEL

Key ideas: portfolio choice, diversification, mutual fund theorem, CAPM

Diversification 3
Dollar portfolios 4
Portfolios with one risky asset 5
Efficient portfolios 7
Mutual fund theorem 8
Pricing individual assets 9
Capital asset pricing rule 12
We have seen that if there are as many linearly independent asset returns as states, asset markets are a perfect substitute for markets in state claims. But what if there are fewer asset markets? In general, trading in asset markets will yield a Pareto-inferior outcome. Moreover, it is generally not possible to solve analytically for the equilibrium asset prices.

One special case in which it is possible to price assets, regardless of whether there are more assets than states, is when individuals only care about the first and second moments of the distribution of asset returns.
Mean-variance preferences \( U^h = U^h (\sigma, \mu) \)

Diversification

A risky assets and one riskless asset (asset 0). Asset \( a \) has return \( z_a = (z_{a1}, \ldots, z_{aS}) \). The vector of mean returns is \( \mu = (\mu_0, \ldots, \mu_A) \) and the covariance matrix of returns is \([\sigma_{ij}]\). The vector of asset prices is \( P = (P_0, \ldots, P_A) \). We choose units of the riskless asset so that a price per unit is 1. Then the gross riskless return is \( \mu_0 = 1 + r \) where \( r \) is the riskless interest rate.

Consumer \( h \), with wealth \( W^h \), chooses a portfolio \( \xi = (\xi_0, \ldots, \xi_A) \) that maximizes the expected utility given her portfolio constraint

\[ P \cdot \xi \leq W^h. \]

Let \( \mu(\xi) \) and \( \sigma(\xi) \) be the resulting portfolio mean and variance, that is

\[ \mu(\xi) = \sum_{a=0}^{A} \xi_a \mu_a \quad \text{and} \quad \sigma^2(\xi) = E \left\{ \sum_{a=0}^{A} \xi_a (z_a - \mu_a) \right\}^2 = \xi' [\sigma_{ij}] \xi. \]

Then the individual maximizes her expected utility by solving the following portfolio problem:

\[ \underset{\xi}{\text{Max}} \{ U^h (\sigma(\xi), \mu(\xi)) \mid P \cdot \xi \leq W^h \}. \]
**Dollar portfolios**

It proves convenient not to focus on the size of an individual’s portfolio, but on spending per dollar of wealth, \( x = \xi / W^h \). The portfolio constraint can then be rewritten as

\[
P \cdot x \leq 1.\]

Note that because \( q = W^h x \), it follows that \( \mu(\xi) = W^h \mu(x) \) and \( \sigma(\xi) = W^h \sigma(x) \). Expected utility can then be expressed as \( U^h(W^h \mu(x), W^h \sigma(x)) \). Rather than carry the wealth term, we define the derived utility function \( u^h(\mu(x), \sigma(x)) \equiv U^h(W^h \mu(x), W^h \sigma(x)) \). Thus, we may rewrite the optimization problem as follows:

\[
Max \{ u^h(\sigma(x), \mu(x)) \mid P \cdot x \leq 1 \}.
\]

We have therefore established that we can analyze the portfolio choice of consumer \( h \) as if she has a wealth of 1.
Portfolios with one risky asset

Consider first an individual choosing a portfolio consisting of the riskless asset (asset 0) and one risky asset (asset 1). Given a wealth of 1, he can purchase 1 unit of the riskless asset or $1/P_1$ units of the risky asset. If he purchases only the riskless asset ($N_0$ in Figure 8.3-1) his mean return is $1+r$. If he purchases only the risky asset ($N_1$ in Figure 8.3-1) the standard deviation and mean of his portfolio are $\mu_1 / P_1$ and $\sigma_1 / P_1$ respectively.

If he spends a fraction $\lambda$ of his wealth on the risky asset, the standard deviation of his portfolio return is $\lambda \sigma_1 / P_1$, and the mean portfolio return is $(1 - \lambda) \frac{\mu_0}{P_0} + \lambda \frac{\mu_1}{P_1}$. Thus the diversified portfolios are all those on the heavy line.

Figure 8.3-1: Choosing an optimal portfolio
**Borrowing and selling short**

If the consumer borrows funds at the riskless interest rate he can purchase more of the risky asset. Equivalently, he can sell short. That is, he can ask his broker to sell units of the riskless asset that others are holding at the brokerage house, use the funds from the sale to buy more of the risky asset and then later pay the riskless gross return (from the sale of the risky asset). Through such a short sale the portfolio mean and standard deviation rise even further along the dashed extension of the line depicted in Figure 8.3-1.
Next define $R$ to be the set of feasible means and standard deviations of an individual who invests only in risky assets. These are depicted in Figure 8.3-2.

Pick any point $(\hat{\sigma}, \hat{\mu})$ in this set. This point depicts the standard deviation and mean of a portfolio $(\hat{x}_1, \ldots, \hat{x}_A)$ costing a dollar.

Think of this as a dollar investment in a mutual fund.

All outcomes on the line $\hat{L}$ are feasible as they are averages on the riskless asset and a single risky asset.

This argument holds for all points in the set $R$. Thus the set of efficient portfolios is the line $\hat{L}^*$ touching the set $R$ at $(\sigma^*, \mu^*)$. Let $(x_1^*, \ldots, x_A^*)$ be the portfolio of risky assets corresponding to this point. Then, if this portfolio is offered by financial intermediaries as a mutual fund, the individual can do no better than choose a portfolio consisting only of the riskless asset and this mutual fund. Of course the optimal share of risky assets (the point $D^*$ in the figure) is determined by both the individual’s wealth and attitude towards risk.
Note that exactly the same argument holds for every investor. Thus, given the asset price vector $P$, every individual will wish to purchase the same mutual fund $(x_1^*,...,x_A^*)$. But the total supply of risky assets is the entire market portfolio. In equilibrium the demand for risky assets (demand for shares in the single mutual fund) must be equal to the total supply of risky assets (the “market portfolio”). Thus, the equilibrium mutual fund must be the market portfolio. Therefore, risk is fully diversified if all individuals trade only in the riskless asset and the market portfolio.

**Proposition 8.3-1: Mutual Fund Theorem**

If individuals care only about the first and second moments of the distribution of portfolio asset returns, then, in equilibrium, no investor can do better than invest in the riskless asset and the market portfolio.
**Pricing individual assets**

Consider a dollar portfolio consisting of the mutual fund $M$, asset $a$ and the riskless asset. The portfolio constraint is

$$x_0 + P_M x_M + P_a x_a = 1.$$ 

From the mutual fund theorem we know that, in equilibrium, the investor will choose only the market portfolio and the riskless asset so that the equilibrium holding of asset $a$ is zero. This is illustrated in Figure 8.3-3. The lightly shaded area indicates all $(\sigma, \mu)$ outcomes when the investor invests in all $A$ risky assets. The dark shaded area indicates the possible outcomes when the investor’s portfolio contains only asset $a$ and the market mutual fund. Because this is a more constrained set of opportunities, the dark shaded area must lie inside the lightly shaded area. Then the dark shaded area must be tangential to the market line.
Therefore if the investor is constrained to choose a portfolio consisting only of the riskless asset, the market mutual fund and asset \( a \), his optimal choice is to purchase only the market mutual fund and the riskless asset. We use this fact to price each individual asset.

If the investor holds only the market portfolio the mean portfolio return per dollar is \( \frac{\mu_M}{P_M} \) and the standard deviation is \( \frac{\sigma_M}{P_M} \). Therefore the slope of the market line (or “Sharpe ratio”) is

\[
\frac{\frac{\mu_M}{P_M} - (1 + r)}{\frac{\sigma_M}{P_M}} = \frac{\mu_M}{\sigma_M} - (1 + r) \frac{P_M}{P_M}
\]

(8.3-1)

Now consider a portfolio consisting of the riskless asset, the market mutual fund, and asset \( a \).

The portfolio return is \( c = x_0 z_0 + x_M z_M + x_a z_a \). Substituting for \( x_0 \) from the portfolio constraint, the portfolio return can be rewritten as follows:

\[
c = 1 + r + x_M (z_M - (1 + r) P_M) + x_a (z_a - (1 + r) P_a).
\]

The mean and variance of this portfolio are therefore as follows:

\[
\mu(x) = 1 + r + x_M (\mu_M - (1 + r) P_M) + x_a (\mu_a - (1 + r) P_a)
\]
\[ \sigma^2(x) = x_M^2 \sigma_M^2 + 2x_M x_a \sigma_{aM} + x_a^2 \sigma_a^2 \]

Differentiating by \( x_a \),

\[ \frac{\partial \mu}{\partial x_a} = \mu_a - (1 + r_0) P_a, \quad \text{and} \quad 2\sigma \frac{\partial \sigma}{\partial x_a} = 2x_M \sigma_{aM} + 2x_a \sigma_a^2 \quad \text{(8.3-2)} \]

Finally, we note that, in equilibrium, the best portfolio of risky assets contains only the market portfolio. Thus, in equilibrium, \( x_a = 0 \) and so \( \sigma(x) = x_M \sigma_M \). Substituting into (8.3-2),

\[
\frac{d \mu}{d \sigma} = \frac{\partial \mu}{\partial x_a} \frac{\partial x_a}{\partial \sigma} = \frac{\mu_a - (1 + r) P_a}{\sigma_{aM} / \sigma_M}
\]

This must be the slope of the market line given by (8.3-1). Therefore

\[
\frac{\mu_a - (1 + r) P_a}{\sigma_{aM} / \sigma_M} = \frac{\mu_M - (1 + r) P_M}{\sigma_M}
\]
Rearranging this expression we have the price of each asset as a function of the underlying means and covariances and the price of the market portfolio. We summarize this in Proposition 8.3-2.

**Proposition 8.3-2: Capital asset pricing rule**

If individuals care only about the first and second moments of the distribution of portfolio asset returns, then the equilibrium price of asset $i$ satisfies

$$
\mu_i - (1 + r_0) P_a = \frac{\sigma_{aM}^2}{\sigma_M^2} \left( \mu_M - (1 + r_0) P_M \right).
$$

(8.3-3)
**Asset yields**

Let $1 + r_a$ be the risky gross yield on asset $a$. That is, $1 + r_a = \frac{Z_a}{P_a}$, $a = 1, \ldots, A$

Then $\sigma_{aM} = \text{cov}(z_a, z_M) = \text{cov}(P_a r_a, P_M r_M) = P_a P_M \text{cov}(r_a, r_M)$.

Similarly, $\sigma_M^2 = \text{var}(z_M) = \text{var}(P_M r_M) = P_M^2 \text{var}(r_M)$.

Substituting into (8.3-3) and rearranging,

$$\frac{\mu_a}{P_a} - (1 + r) = \frac{\text{cov}(r_a, r_M)}{\text{var}(r_M)} \left( \frac{\mu_M}{P_M} - (1 + r) \right).$$

Hence

$$E\{r_a\} - r = \frac{\text{cov}(r_a, r_M)}{\text{var}(r_M)} (E\{r_M\} - r).$$

Investment houses run regressions of each stock’s yield on the market yield and report the “beta” of the stock. Suppose two listed firms have the same expected yield, but the return of the first firm has a higher beta (is more highly correlated with the market portfolio). Then the equilibrium expected yield for the first firm must be higher because it offers less of an opportunity to spread risk.