ANSWERS TO EVEN NUMBERED EXERCISES IN CHAPTER 9

SECTION 9.1: SIMULTANEOUS “ONE SHOT” GAMES

Exercise 9.1-2 Vickrey Bidding Game

Bidder 1 values an item for sale at 7 (thousand) and bidder 2 at 8 (thousand). Each must submit a sealed bid of 5 thousand or higher in increments of 1 thousand. Since no one values the item above 8 we consider only bids up to and including 8. The bidder who bids the most is the winner but only has to pay the second highest bid. If the bids are the same, the winner is selected by a coin toss and the winner pays the common bid. Bids are submitted in secret.

(a) Explain why the payoff matrix must be as depicted below.

<table>
<thead>
<tr>
<th>Bidder 1</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bidder 2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1, 1.5</td>
<td>0, 3</td>
<td>0, 3</td>
<td>0, 3</td>
</tr>
<tr>
<td>6</td>
<td>2, 0</td>
<td>0.5, 1</td>
<td>0, 2</td>
<td>0, 2</td>
</tr>
<tr>
<td>7</td>
<td>2, 0</td>
<td>1, 0</td>
<td>0, 0.5</td>
<td>0, 1</td>
</tr>
<tr>
<td>8</td>
<td>2, 0</td>
<td>1, 0</td>
<td>0, 0</td>
<td>-0.5, 0</td>
</tr>
</tbody>
</table>

Fig. 9.1-10: Vickrey auction

(b) Show that (5,8) and (6,8) and (7,8) are all Nash equilibria.

(c) Are there other equilibria in pure strategies?

(d) Show that one of the Nash equilibria is an equilibrium in weakly dominant strategies.

ANSWER

(a) If there is a tie (as is the case on the diagonal) each wins with probability ½. The payoff to the winner is his valuation less his bid. Thus his expected payoff is half his payoff if he wins. Off the diagonal the high bidder wins so his payoff is his valuation less the lower bid.
(b) If bidder 1 chooses 5, it is a best response for bidder 2 to choose 8. And given that bidder 2 chooses 8, bidding 5 is a best response. An identical argument holds for the other two strategy pairs.

(c) If bidder 1 bids 5, then 6 and 7 are also best responses for bidder 2. If bidder bids 7, bidder 1’s best response is to choose 5 so (5,7) is another Nash Equilibrium. Similarly (6,7) and (8,5), (8,6) and (8,7) are Nash Equilibria.

(d) For bidder 2 the last column weakly dominates the other three columns. For bidder 1 the third row weakly dominates the other three rows. Thus (7,8) is the unique equilibrium in weakly dominant strategies.

**Exercise 9.1-4: Odds and Evens**

Consider the following simplified version of the widely used “Odds and Evens” game. Each player simultaneously holds up one or two fingers and calls either “Odds” or evens. If the number of fingers is Odd the player calling “odds” wins. If the total is even the player calling “Even” wins. If both make the same call no one wins. In this case an action is a number of fingers and a call. Thus each player has four possible actions. The payoff matrix is shown below.

<table>
<thead>
<tr>
<th></th>
<th>O1</th>
<th>O2</th>
<th>E1</th>
<th>E2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p_i)</td>
<td>O1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>0,0</td>
<td>0,0</td>
<td>-1,1</td>
</tr>
<tr>
<td>(q_i)</td>
<td>O2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>0,0</td>
<td>0,0</td>
<td>1,-1</td>
</tr>
<tr>
<td>(r_i)</td>
<td>E1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>1,-1</td>
<td>-1,1</td>
<td>0,0</td>
</tr>
<tr>
<td>1- (p_i - q_i - r_i)</td>
<td>E2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>-1,1</td>
<td>1,-1</td>
<td>0,0</td>
</tr>
</tbody>
</table>

**Fig. 9.1-11: Odds and Evens game**

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1 This game was played by the Romans. We have simplified it here by having only one round of the game. Typically the game continues until one or the other player is the winner.
(a) Establish that there are no weakly dominated strategies and no NE in pure strategies.

(b) Show that there is a continuum of NE in completely mixed strategies (all strategies played with strictly positive probability).

(c) Show that there are at least 6 NE in which each player plays 2 strategies with positive probability.

(d) Use your result from (b) to draw a conclusion about a NE of the standard Odds and Evens game where play continues until there is a winner.

**ANSWER**

(a) No weakly dominated strategy

For player 1 the lowest payoff for any pure strategy $s_1$ is -1. This occurs if player 2 makes an opposite call and the total number of fingers is even. Let $s_1'$ be the strategy where the number of fingers player 1 raises is changes. Player 1 now gets a payoff of 1. Thus $s_1$ does not weakly dominate $s_1'$. A symmetric argument holds for player 2.

No NE in pure strategies

For any pure strategy of player 1, player 2 has a unique best response in which his payoff is +1. Then player 1’s payoff is -1. Given this best response, it follows from the previous argument that player 1 is strictly better off changing the number of fingers. Thus there is no pair of pure strategies $s_i \in S$ that are mutual best responses.

(b) Given the symmetry of the game is a NE for both players to play the mixed strategy $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. Suppose that player 2’s mixed strategy is $(\alpha, \beta, \gamma, \delta)$. The payoffs for player 1 are

<table>
<thead>
<tr>
<th>Strategy</th>
<th>$E{u_1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>O1</td>
<td>$(\delta_2 - \gamma_2)$</td>
</tr>
<tr>
<td>O2</td>
<td>$-(\delta_2 - \gamma_2)$</td>
</tr>
<tr>
<td>E1</td>
<td>$(\alpha_2 - \beta_2)$</td>
</tr>
<tr>
<td>E2</td>
<td>$-(\alpha_2 - \beta_2)$</td>
</tr>
</tbody>
</table>
Suppose at least one payoff is strictly positive. WOLOG suppose $\delta > \gamma$ and $\alpha \geq \beta$. Then player 1 will never chose anything but O1 and E1 and so the best response is not completely mixed. Then all payoffs must be zero and so

$$\delta - \gamma = \alpha - \beta = 0.$$  

A symmetrical argument hold for player 2.

The new payoff matrix and mixed strategies are therefore as shown below.

\[
\begin{array}{cccc}
\text{Agent 2} & \alpha_2 & \alpha_2 & \frac{1}{2} - \alpha_2 & \frac{1}{2} - \alpha_2 \\
O1 & 0,0 & 0,0 & -1,1 & 1,-1 \\
O2 & 0,0 & 0,0 & 1,-1 & -1,1 \\
E1 & 1,-1 & -1,1 & 0,0 & 0,0 \\
E2 & -1,1 & 1,-1 & 0,0 & 0,0 \\
\end{array}
\]

Fig. 9.1-11: Equilibrium in completely mixed strategies.

Expected payoffs are zero for all strategies. Thus the depicted strategies are mutual best responses. The strategies are completely mixed for all $\alpha_i \in (0, \frac{1}{2})$, $i = 1, 2$.

(c) For any $(\alpha_1, \alpha_2) \in \{(0,0), (0, \frac{1}{2}), (\frac{1}{2},0), (\frac{1}{2}, \frac{1}{2})\}$, the argument above establishes that these are NE strategies. It is easily confirmed that $(\frac{1}{2},0, \frac{1}{2},0)$ and $(0, \frac{1}{2}, 0, \frac{1}{2})$ are also NE mixed strategies.

(d) It is natural to focus on the symmetric equilibrium. Instead of the game ending if both call “Odds” or both call “Evens,” the game continues. If the symmetric equilibrium is used in all future rounds the continuation payoff is zero. Thus the payoff matrix on the one round game is the reduced form of the multi round game where it is believed that both players will use the symmetric completely mixed strategy in all future rounds.
SECTION 9.2: MULTI-ROUND GAMES

Exercise 9.2-2: Alternating offer game
A pie of value \( v \) is to be divided between two players, but only if they agree on how much each should get. In each round one player makes a demand and the other either accepts or rejects the offer. Player 1 moves first and makes a demand of \( x_1 \) in round 1. Then player 2 has \( v - x_1 \). If the offer is accepted the game ends. If the offer is rejected player 2 demands \( x_2 \) and so on. Between each round one quarter of the pie is lost. Thus after 4 rounds there is nothing left.

(a) Assume that if a player is indifferent he or she will accept the other’s demand with probability 1. Consider the sub-game beginning in round 4. Only one quarter of the pie remains. If player 2 makes any offer \( x_4 < \frac{1}{4} v \) it will be accepted by player 1 since this gives player 1 \( u_{i4} = \frac{1}{4} v - x_4 > 0 \) and if he rejects there is no pie left. By hypotheses a player accepts if he or she is indifferent so the best response of player 2 is \( x_4 = \frac{1}{4} v \). Apply backwards induction to show that the initial offer will be \( \frac{1}{2} v \) and that it will be accepted.

(b) Show that if player 1 accepts in the last stage with probability \( p_4 < 1 \) there is no best response by player 1. Use this argument to establish that the equilibrium of part (a) is the unique sub-game perfect equilibrium.

(d) For any integer \( T \), solve for the sub-game perfect equilibrium payoff if \( v / T \) of the pie is lost each round.

(e) What Nash equilibrium is most favorable to player 1? What Nash equilibrium is most favorable to player 2.

ANSWER

(a) In any odd round the strategies of the two players can be written as 
\( (s_1', s_2') = (x_i, p_i) \) where \( p_2 \) is the probability that player 2 accepts. In even numbered rounds the strategies can be written as \( (p_i, x_i) \). Consider round 4. There is no best offer \( x_4 \) less than \( \frac{1}{4} v \). So suppose \( x_4 = \frac{1}{4} v \). Player 1 is indifferent so a best response is to accept, that is \( p_4 = 1 \). Then the sub-game perfect equilibrium payoffs in round 4 are...
(u_{1t}, u_{2t}) = (0, \frac{1}{4}v)$. Making the same argument in round three, the sub-game perfect equilibrium payoffs are $(u_{1t}, u_{2t}) = (\frac{1}{4}v, \frac{1}{4}v)$. Continuing to work backwards, the equilibrium payoffs are $(u_{11}, u_{21}) = (\frac{1}{2}v, \frac{1}{2}v)$.

(b) Consider the round 4 strategy pair $(p_1, \frac{1}{4}v)$ where $p_4 < 1$. These are not mutual best responses. Note that player 2’s payoff is $\frac{1}{4}pv$. If player 2 demands $x_4 < \frac{1}{4}v$ it will be accepted. Thus player 2 is strictly better off choosing $x_4$ sufficiently close to $\frac{1}{4}v$. But there is no best response $x_4(p_1)$ for $p_1 < 1$.

**Exercise 9.2-4: To take or not to take, that is the question**

Two participants, $A$ and $B$ have agreed to pay a game at a well known casino. Any winnings will be in the form of a cashier’s check, to be mailed the following day. The game is as follows. A prize of $500 will be offered to $A$. If he takes it the game will be over. $A$ is also told that if he declines, then after he leaves, a total of $1000 + x$ will be offered to $B$. She can either take it or decline. If $B$ also declines each will be sent a check for $1000$.

(a) If $x > 0$ explain why $A$’s unique SPE is for each player to take the money.

(b) Suppose $(i) x = 500 \quad (ii) x = 2000$. If you were contestant $B$ what would you do in each case?

**ANSWER**

(a) Consider the sub-game played by $B$. For $x > 0$, $B$’s best response is take the money. Then at the first stage of the game player $A$’s best response is take the money.

(b) Many people are distrustful if they are player $A$. But if player $A$ does choose to decline the money, a sense of fairness often leads to $B$ choosing to decline as well. This sense of fairness is more than offset if the payoff to taking the money is sufficiently high.

(c) If player $B$ has to face the person she cheats, she is more likely to share.

(d) While some find this a major refutation of the theory of games, most economists argue that such case are relatively rare and when they do occur it is possible to incorporate the disutility of embarrassment into the payoffs.
SECTION 9.3: DUOPOLY GAMES

Exercise 9.3-2: Cournot production game
Suppose that the market demand curve is \( p = a - q \). There are two firms. The output of firm j is \( q_j \). Each has the same unit cost of production \( c \). The game is played once.

(a) Solve for the Nash Equilibrium and compare it with the symmetric collusive outcome.

(b) For \( n \) identical firms show that the first order condition for a best response is

\[
\frac{\partial \Pi}{\partial q_i} = a - c - b(s + q_i) = 0, \text{ where } s = \sum_{j=1}^{n} q_j.
\]

Sum over the \( n \) firms and hence show that \( n(a - c) - (n + 1)s = 0 \)

Hence show that the equilibrium price is \( p^n = \frac{1}{n+1} a + \frac{n}{n+1} c \).

(d) Compare the outcome with the Walrasian equilibrium outcome as \( n \) becomes large.

ANSWER

(a) The profit of firm 1 is \( U_1(q) = (a - c - q_1 - q_2)q_1 = (a - c - q_2)q_1 - q_1^2 \). Given \( q_2 \), firm 1 has a best reply \( q_1^{BR} = \arg \max_{q_1} U_1(q) \). Solving, \( q_1^{BR} = \frac{1}{2}(a - c - q_2) \). Arguing symmetrically for firm 2, \( q_2^{BR} = \frac{1}{2}(a - c - q_1) \).

For a Nash Equilibrium both strategies must be best responses. Solving these two equations, \( q_j^N = \frac{1}{2}(a - c) \) hence \( U_j^N = \frac{1}{8}(a - c)^2 \)

Total profit is

\[ U_1 + U_2 = (p - c)q = (a - c - q)q. \]

This takes on its maximum at \( q = (a - c)/2 \). Thus if firms share equally, the collusive profit of firm j is \( U_j^C = \frac{1}{8}(a - c)^2 \).

(b) Let \( s = \sum_{i=1}^{n} q_i \) be the total output of the \( n \) firms. The profit of firm j is

\[ U_j(q) = (a - c - s)q_j = (a - c - \sum_{i \neq j} q_i)q_j - q_j^2. \]

Firm j chooses \( q_j \) to maximize \( U_j(q_j, q_{-j}) \). Differentiating, the FOC is
\[
\frac{\partial}{\partial q_j} U_j(q) = (a - c - \sum_{i \neq j} q_i) - 2q_j = 0.
\]

Rearranging,
\[
q_j = (a - c - s), \quad j = 1, \ldots, n.
\]

Summing over the n firms, \( s = n(a - c - s) \) hence total output, \( s = \frac{n}{n+1}(a - c) \) and the equilibrium price is \( p = a - s = \left(\frac{1}{n+1}\right)a + \left(\frac{n}{n+1}\right)c.\)

**Exercise 9.3-4: Production game with sequential entry**

Demand is given by \( p = 24 - q \), where \( q = q_1 + q_2 \). Firm 1 has a cost \( C_1 = 4q_1 \). Firm 2’s cost function is \( C_2 = 8q_2 \).

(a) Suppose that each firm makes its production decision without observing its competitor’s decision. Solve for the best response functions for each firm. Hence solve for the simultaneous move equilibrium.

(b) Suppose firm 1 makes the first move. Utilize the fact that firm 2 will choose a best response to get an expression for firm 1’s profit in terms of \( q_1 \) alone.

(c) Solve for firm 1’s optimal output decision and compare output levels and profits with those in the simultaneous move equilibrium.

**ANSWER**

(a) Firm \( j \) chooses \( q_j \) to maximize its profit \( U_j(q) = (24 - c_j - q_1 - q_2)q_j \). The FOC is
\[
\frac{\partial U_j}{\partial q_j} = p(q) - c_j + \frac{\partial p}{\partial q_j}q_j = p(q) - c_j - q_j = 0.
\]

It follows that \( q_1 + c_1 = q_2 + c_2 \) hence \( q_1 = q_2 + 4 \). Substituting back into the FOC for commodity 1 if follows that \( q^N = (8, 4) \).

(b) From part (a), firm 2 chooses \( q_2 \) to satisfy
\[
\frac{\partial U_2}{\partial q_2} = p(q) - 8 - q_2 = 16 - q_1 - 2q_2 = 0.
\]
Thus \( q_2^{BR} = 8 - \frac{1}{2} q_1 \). Then

\[
U_1(q_1) = (24 - c_1 - q_1 - q_2^{BR}(q_1))q_1 = (12 - \frac{1}{2} q_1)q_1
\]

(c) Firm 1 chooses \( q_1 \) to maximize \( U_1 \). From the FOC, \( q_1^N = 12 \), hence \( q_2^N = 2 \)

Total output is higher then in the simultaneous move game thus the equilibrium price is lower. Firm 1 must be better off moving second since one of its feasible alternatives is its choice in the simultaneous move game.

**Exercise 9.3-6: Existence of equilibrium**

Consider two firms producing an identical product. The unit cost for firm \( i \) is \( c_i \) where \( c_1 < c_2 \). All consumers purchase from the firm offering the lowest price. If the two firms set the same price firm 1 gets a fraction \( f_1 \) of the market and firm 2 a fraction \( f_2 = 1 - f_1 \).

Consumer demand for the two products is as follows.

\[
(q_1, q_2) = \begin{cases} 
(a - p_1, 0), & p_1 < p_2 \\
(f_1(a - p_1), f_2(a - p_2)), & p_1 = p_2 \\
(0, a - p_2), & p_1 > p_2 
\end{cases}
\]

The action set for each firm is the interval \([0, a]\).

(a) Show that there can be no equilibrium in pure strategies with either price strictly greater than \( c_2 \).

(b) Suppose that firm 1 adopts a mixed strategy. Let \( p_1^* \) be the firm’s maximum price . That is, \( \mu_i(p_1^*) = 1 \) and \( \mu_i(p_1) < 1 \), for all \( p < p_1^* \). Suppose that \( p_1^* > c_2 \) and obtain a contradiction.

HINT: Argue that firm 2’s best response is to announce a price \( p_2 < p_1^* \) and hence that the price \( p_1^* \) cannot be a best response for firm 1.

(c) Show that \((p_1, p_2) = (c_2, c_2)\) is the unique Nash equilibrium if \( f_1 = 1 \).
(d) Show that there is no Nash equilibrium if \( f'_1 < 1 \).

(d) For finite strategy sets show that there is a unique equilibrium.

(e) Do you think it makes sense to model Bertrand price competition by assuming that \( f'_1 = 1 \)?

**ANSWER**

(a) If \( p_1 > c_2 \) firm 2 splits the market by matching and doubles its market share by slightly under pricing. Thus under pricing is always more profitable.

(b) If firm 1 chooses its maximum price \( p_1^* > 1 \) with strictly positive probability the argument of (a) applies. The market share rises discontinuously as firm 2 lowers its price below \( p_1^* \). Suppose instead that \( \mu(p_1) \) is continuous at \( p_1^* \). Then firm 2 wins with zero probability if it chooses \( p_2 = p_1^* \) and with strictly positive probability if it chooses \( p_2 < p_1^* \). Then firm 1 should choose a price less than or equal to \( p_2 \).

(c) We have ruled out any equilibrium with prices above \( c_2 \). If \( (p_1, p_2) = (c_2, c_2) \) the profits are \( (u_1, u_2) = ((a - c_2)c_2, 0) \). Firm 2 loses money by lowering prices. Firm 2 lowers its profit by lowering prices.

(d) We have ruled out any equilibrium with prices above \( c_2 \). If \( (p_1, p_2) = (c_2, c_2) \) the profits are \( (u_1, u_2) = (f_1(a - c_2)c_2, 0) \). Firm 2 loses money by lowering prices. If firm 2 lowers its price to \( p_1 = c_2 - \delta \) its profit is \( u_1 = (a - c_2 - \delta)(c_2 - \delta) = (a - c_2)c_2 - \delta(c_2 - \delta) \). For small \( \delta \) this strictly exceeds \( f_1(a - c_2)c_2 \). However there is no best response since \( \delta \) can be any strictly positive real number.
(e) Suppose that $c_2$ is in the finite strategy set of both firms. Arguing as in (d), if the price differences are sufficiently small, the equilibrium strategy of firm 1 is to choose the highest price. For the case $p_1 = p_2 = c_2$, firm 2 could purchase from firm 1 rather than produce to order. Then, while firm 1’s final market share is $f_1$, its total supply is $q = a - c$.

Exercise 9.3-8: Rent seeking contest

Two competing firms spend resources on a politician. The amount spent by firm $i$ is $x_i$. The firm that spends the most on the politician’s re-election wins a favor from the politician. If firm 1 wins the contest, its profit rises by $W_1$ and the profit of firm 2 falls by $L_2$. If firm 2 wins, its profit rises by $W_2$ and the profit of firm 1 falls by $L_4$. If the two firms spend the same amount, each wins with probability 0.5. The profit gains and losses are common knowledge. Firms choose their spending levels simultaneously. A firm may also choose not to participate in which case this firm has a zero payoff and the other firm wins with any bid.

(a) First consider the special case $(W_i, L_i) = (W, 0)$ $i = 1, 2$. Give an argument as to why there can be no equilibria in pure strategies.

(b) Show that the mixed strategy with probability distribution $\mu_i(x) = \Pr\{x_i \leq x\} = x_i / W$ is a Nash equilibrium.

(c) For the general model sketch an argument as to why the equilibrium probability distributions $\mu_i(x), \mu_2(x)$ cannot have a mass point at $x > 0$.

(d) Solve for the equilibrium mixed strategy if $(W_i, L_i) = (W, L) i = 1, 2$. 

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(e) Solve for the equilibrium mixed strategies if $W_1 > W_2 > L_1 = L_2 = 0$

**ANSWER**

(a) Consider the pure strategy profile $(x_1, x_2)$ where $x_1 < x_2 < W$. Firm 1 is better bidding just above $x_2$ than bidding below $x_2$. Thus the strategies are not mutual best responses. Next consider $(x_1, x_2)$ where $x_1 < W \leq x_2$. Firm 2’s bid is not a best response because it can bid strictly lower than $W$ and make a profit. Since a symmetrical argument holds for firm 2 it follows that there is no equilibrium in pure strategies with $x_1 \neq x_2$.

Suppose next that $x_1 = x_2 = \bar{x} < W$. The two firms make a profit with probability $\frac{1}{2}$. But then firm 1 can increase its expected profit by increasing its bid to $\bar{x} + \delta$ where $\delta$ is positive and sufficiently small. The expected profit rises from $\frac{1}{2}W - x_1$ to $W - x_1 - \delta$.

Finally suppose that $x_1 = x_2 = \bar{x} \geq W$. Both firms have an expected loss. Now firm 1 is strictly better off lowering its bid to zero.

(b) All we need to do is check that the mixed profiles are best responses. If firm 1 bids $x_i$ it wins if $x_2 < \bar{x}_i$ and wins with probability $\frac{1}{2}$ if $x_2 = x_i$. Firm 1’s win probability is therefore

$$\Pr\{x_2 \leq x_i\} = \frac{x_i}{W}.$$  

Firm 1’s expected profit is therefore

$$u_1(x) = \Pr\{\text{firm 1 wins}\} W - x_i = 0, \quad x_i \in [0,W].$$

Thus firm 1 is indifferent between all bids on $[0,W]$.

(c) If firm 1 bids $x_i$ with probability $\pi$, firm 2’s payoff rises discontinuously at $x_2 = \hat{x}_i$.  

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Thus there is some interval \((\hat{x}_i - \delta, \hat{x}_i]\) over which firm 2 will not bid. But then firm 1’s win probability is constant over \((x_i - \delta, x_i]\) and so her payoff is a strictly deceasing function of \(x_2\) over this interval. This contradicts our initial assumption that bidding \(\hat{x}_i\) with probability \(\pi > 0\) is a best response.

(d) Suppose that firm 1 bids \(x_i \geq 0\) with probability \(G_i(x_i)\) where \(G\) is continuous for all \(x_i > 0\). If firm 2 chooses \(x_2 > 0\) her win probability is \(G(x_2)\). Hence her expected payoff is

\[
u_2(x_2) = G_i(x_2)W - (1-G(x_2)L - x_2 = G_i(x_2)(W+L) - L - x_2
\]

\[
= (W+L)(G_i(x_2) - \frac{L+x_2}{W+L}).
\]

For an equilibrium in continuous mixed strategies this must be constant. Hence

\[
G_i(x_2) = \frac{L+x_2}{L+W}, \text{ and } \nu_2(x_2) = 0, \ x_2 > 0
\]

Suppose firm 1 does not bid with probability \(\frac{L}{L+W}\). Then

\[
u_i(0) = (\frac{L}{L+W})W - (1-\frac{L}{L+W})L = 0.
\]

Then firm 2 is indifferent between not competing and making any bid on \([0, W+L]\).

(e) Suppose that firm \(j\) bids \(x_j \geq 0\) over \([0, \beta]\) with probability \(G_j(x_j)\) where \(G\) is continuous for all \(x_j > 0\). If firm 2 chooses \(x_2 > 0\) her win probability is \(G(x_2)\). Hence her expected payoff is

\[
u_2(x_2) = G_i(x_2)W_2 - x_2 = W_2(G_i(x_2) - \frac{x_2}{W_2})
\]

Similarly,

\[
u_i(x_i) = G_j(x_i)W_1 - x_i = W_1(G_j(x_i) - \frac{x_i}{W_1})
\]

Since \(G_i(\beta) = G_j(\beta) = 1\) it follows that \(u_i(\beta) > u_2(\beta)\). To compete, firm 2 must have a positive expected payoff. Therefore \(u_i(\beta) > 0\). It follows that

\[
u_i(0) = W_1G_j(0) = u_i(\beta) > 0.
\]
Arguing as above, there cannot be a tie at zero with positive probability. Therefore $G_t(0) = 0$ and so $u_2(0) = 0$. Then $u_2(\beta) = W_2 - \beta = 0$ and so $G_2(x_2) = \frac{x_2}{W_2}$.

Also $u_1(x_1) = W_1 G_2(x_1) - x_1 = W_1 - \beta = W_1 - W_2$. Therefore $G_1(x_1) = 1 - \frac{W_2}{W_1} + \frac{x_1}{W_1}$.

Note that the bigger the asymmetry, the higher the probability that firm 2 does not compete and hence the higher the equilibrium payoff to the high value firm.

**SECTION 9.4: INFINITELY REPEATED GAMES**

*Exercise 9.4-2: Nash equilibrium threats*

For the data of the previous exercise, consider the strategy $q^* = (3,3)$. Player 1 can compute his opponent’s maximum payoff for each of his actions, $\max_{q_2} u_2(q_1,q_2)$. Let $q_1^M$ be the strategy that minimizes player 2’s maximum payoff.

$$q_1^M = \arg \max_{q_1} \min_{q_2} u_2(q_1,q_2)$$

This is known as player 1’s Minmax strategy. In this example, if player 1 chooses $q_1 \geq 12$, player 2 can do no better than produce nothing. Therefore player 1 can drive player 1’s payoff to zero. Of course the same argument holds for player 2.

(a) Explain why the Minmax threat by both players is a Nash equilibrium threat but not a SPE threat.

(b) For what discount factors is $q^* = (3,3)$ a Nash equilibrium outcome of the infinitely repeated game?

**ANSWER**

(a) The proposed strategy is to play the cooperative strategy in period 1 and continue to play this unless the other player defects. If he does so the planned response is to produce 12 units in each future period.

Consider the efficient cooperative strategy $q = (3,3)$ so that $u_1^C = 18$

(b) If player 2 adopts this strategy and player 1 defects in the first round, player 1’s first period payoff is $u_1^D = u_1(q_1^D,x) = \frac{1}{4} (12 - x)^2 = 81/4$. After that player 2 chooses output
so that price is below unit cost if player 1 produces any output. So player 1’s best response is to produce zero. Then the present value of his payoff stream is $81/4$. If he cooperates forever his payoff is $u_i^C/(1-\delta) = 18/(1-\delta)$. Thus firm 1 gains by defecting only if

$$\frac{81}{4} > \frac{18}{1-\delta},$$

that is $\delta < 1/9$.

(c) In a Nash Equilibrium there are no restrictions on behavior off the equilibrium path. In particular we do not need to consider what is player 2’s best response if player 1 defects. For sub-game perfection we require best responses off the equilibrium path. If player 2 produces nothing, player 1’s best response is to produce the monopoly output of 6 rather than the zero profit output of 12.