**Topic 8: Choice over time**

One sector growth model with linear technology  
Life cycle consumption and savings  
Evolution of total wealth  
One sector macro model with concave production function  
Infinite horizon optimization problems  
Malinvaud (sufficient) condition
One sector macro growth model

One representative agent. Economy last for $T$ periods. Initial capital $k_0$. Let $\{c_t\}_{t=1}^T$ be the consumption sequence. Utility is $U(c) = u(c_1) + \delta u(c_2) + \ldots$. Where $u$ is concave. We will focus on the special case $u'(c_t) = \frac{1}{c_t^{1/\sigma}}$. Then $U$ is concave. The depreciation rate is $\beta$.

Output $y_t = \gamma k_t$ is either invested or consumed $c_t + i_t = y_t = \gamma k_t$.

Capital growth equation:

$$k_{t+1} = k_t - \beta k_t + i_t = k_t - \beta k_t + \gamma k_t - c_t = (1 + \theta) k_t - c_t$$

where $1 + \theta = \gamma - \beta$

How should we proceed?
One sector macro growth model

One representative agent. Economy lasts for $T$ periods. Initial capital $k_1$ Let $\{c_t\}_{t=1}^T$ be the consumption sequence. Utility is $U(c) = u(c_1) + \delta u(c_2) + \ldots$. Where $u$ is concave. We will focus on the special case $u'(c_t) = \frac{1}{c_t^{1/\sigma}}$. Then $U$ is concave. The depreciation rate is $\beta$.

Output $y_t = \gamma k_t$ is either invested or consumed $c_t + i_t = y_t = \gamma k_t$

Capital growth equation:

$k_{t+1} = k_t - \beta k_t = i_t = (1 + \theta)k_t - c_t$ where $1 + \theta = \gamma - \beta$

How should we proceed? Consider $T=2$

Constraints

$k_2 - (1 + \theta)k_1 - c_1 \geq 0 \quad k_2 - (1 + \theta)k_1 - c_1 \geq 0$ (We know $k_3^* = 0$)

Thus the vector of choice variables is $x = (c_1, c_2, k_2)$. 
Form the Lagrangian

\[ \mathcal{L}(c, k, \lambda) = f(x) + \lambda \cdot h(x)L = u(c_1) + \delta u(c_2) + \lambda_1 (k_1(1 + \theta) - c_1 - k_2) + \lambda_2 (k_2(1 + \theta) - c_2 - k_3) \]

The Lagrangian is concave in \( x \). We seek a solution to the FOC with all shadow prices strictly positive and consumption strictly positive in each period.

*
Form the Lagrangian
\[
\mathcal{L}(c, k, \lambda) = f(x) + \lambda \cdot h(x)L = u(c) + \delta u(c) + \lambda_1 (k_1 (1 + \theta) - c_1 - k_2) + \lambda_2 (k_2 (1 + \theta) - c_2 - k_3)
\]

The Lagrangian is concave in \(x\). We seek a solution to the FOC with all shadow prices strictly positive and consumption strictly positive in each period.

FOC

\[
\frac{\partial \mathcal{L}}{\partial c_1} = u'(c_1) - \lambda_1 = 0 \quad \Rightarrow \quad \frac{1}{c_1^{1/\sigma}} = \lambda_1 \quad \text{so} \quad c_1 = \left(\frac{1}{\lambda_1}\right)^\sigma \quad (1)
\]

\[
\frac{\partial \mathcal{L}}{\partial c_2} = \delta u'(c_2) - \lambda_2 = 0 \quad \Rightarrow \quad \frac{\delta}{c_2^{1/\sigma}} = \lambda_2 \quad \text{so} \quad c_2 = \left(\frac{\delta}{\lambda_2}\right)^\sigma \quad (2)
\]

\[
\frac{\partial \mathcal{L}}{\partial k_2} = -\lambda_1 + (1 + \theta) \lambda_2 = 0 \quad \Rightarrow \quad \frac{\lambda_1}{\lambda_2} = 1 + \theta \quad (3)
\]

Combining (1) and (2), \(\frac{c_2}{c_1} = \left(\frac{\lambda_1}{\lambda_2}\right)^\sigma\). Substituting from (3) \(\frac{c_2}{c_1} = ((1 + \theta)\delta)^\sigma\).
We have shown that \( \frac{c_2}{c_1} = ((1 + \theta)\delta)^\sigma \). \hspace{1cm} (4)

We will assume that \( (1 + \theta)\delta > 1 \) so consumption increases over time.

It will prove convenient to define the parameter \( \alpha \) to satisfy the following equation.

\[
(1 + \theta)\alpha = ((1 + \theta)\delta)^\sigma
\]

Then

\[
\frac{c_2}{c_1} = (1 + \theta)\alpha \hspace{1cm} (5)
\]

Note that as long as \( \sigma < 1 \), \( \alpha < 1 \). However if \( \sigma \) is sufficiently large \( \alpha \) must exceed 1.
The final step is to solve for first period consumption. To do this we note that the two constraints can be rewritten as follows.

\[ k_2 = (1 + \theta)k_1 - c_1 \quad k_2 = (1 + \theta)k_1 - c_1 \]

\[ k_3 = (1 + \theta)k_2 - c_2 \quad \frac{k_3}{1 + \theta} = k_2 - \frac{c_2}{1 + \theta} \]

Adding:

\[ \frac{k_3}{1 + \theta} = (1 + \theta)k_1 - c_1 - \frac{c_2}{1 + \theta}. \]

But \( k_3 = 0 \). Therefore

\[ c_1 + \frac{c_2}{1 + \theta} = k_1 (1 + \theta). \]

Substituting from (5)

\[ c_1 + c_1 \alpha = c_1 (1 + \alpha) = k_1 (1 + \theta) \quad (6) \]

Then \( c_1 = \frac{1 + \theta}{1 + \alpha} k_1. \)
With three periods (6) becomes
\[ c_1 + c_1\alpha + c_1\alpha^2 = c_1(1 + \alpha + \alpha^2) = (1 + \theta)k_1 \]

With \( T \) periods (6) becomes
\[ c_1 + c_1\alpha + c_1\alpha^2 + \ldots + c_1\alpha^{T-1} = c_1(1 + \alpha + \alpha^2 + \ldots + \alpha^{T-1}) = (1 + \theta)k_1 \]

\[ S_T = 1 + \alpha + \ldots + \alpha^{T-1} = \frac{1 - \alpha^T}{1 - \alpha}. \]  

Hence
\[ c_1\left(\frac{1 - \alpha^T}{1 - \alpha}\right) = (1 + \theta)k_1. \]  That is \( c_1 = \left(\frac{1 - \alpha}{1 - \alpha^T}\right)(1 + \theta)k_1 \)

What if the horizon is infinite?

(i) \( \alpha < 1 \), \( c_1 = \left(\frac{1 - \alpha}{1 - \alpha^T}\right)(1 + \theta)k_1 \)  
(ii) \( \alpha > 1 \), \( c_1 = \left(\frac{\alpha - 1}{\alpha^T - 1}\right)(1 + \theta)k_1 \)
Life-cycle consumption and saving

An individual lives for $T$ periods. He has an initial wealth $K_1$ and has an income sequence of $\{y_t\}_{t=1}^T$. His utility is $U(c) = u(c_1) + \delta u(c_2) + \ldots$. Where $u'(c) = 1 / c^{1/\sigma}$ so that $u$ is concave. The individual can borrow or lend at the interest rate $r$. He wishes to leave a bequest in period $T+1$ of $\bar{k}_{T+1}$.

What should he do?
**Life-cycle consumption and saving**

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What should he do?

Arguing as in the macro model we need to write down the period $t$ budget constraint.

$$k_{t+1} \leq (k_t + y_t - c_t)(1 + r) \quad \text{that is} \quad (k_t + y_t - c_t)(1 + r) - k_{t+1} \geq 0$$

The Constraint Qualification holds as long as $(k_1, y_1) > 0$

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Life-cycle consumption and saving

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Form the Lagrangian

$$\mathcal{L}(c, k, \lambda) = \sum_{t=1}^T \delta^{t-1} v(c_t) + \sum_{t=1}^T \lambda_t ((k_t + y_t - c_t)(1 + r) - k_{t+1})$$

We seek a solution to the FOC with all shadow prices strictly positive and consumption strictly positive in each period. Why?
\[ \mathcal{L}(c, k, \lambda) = \sum_{t=1}^{T} \delta^{t-1} u(c_t) + \sum_{t=1}^{T} \lambda_t ((k_t + y_t - c_t)(1 + r) - k_{t+1}) \]

\[ = ... + \delta^{t-1} u(c_t) + \delta^{t} u(c_{t+1}) + \lambda_t ((k_t + y_t - c_t)(1 + r) - k_{t+1}) + \lambda_{t+1} ((k_{t+1} + y_{t+1} - c_{t+1})(1 + r) - k_{t+2}) + ... \]

FOC

\[ \frac{\partial \mathcal{L}}{\partial c_t} = u'(c_t) - (1 + r) \lambda_t = 0 \quad \Rightarrow \quad \frac{\delta^{t-1}}{c_t^{1/\sigma}} = (1 + r) \lambda_t \quad (1) \]

\[ \frac{\partial \mathcal{L}}{\partial c_{t+1}} = u'(c_{t+1}) - (1 + r) \lambda_{t+1} = 0 \quad \Rightarrow \quad \frac{\delta^{t}}{c_{t+1}^{1/\sigma}} = (1 + r) \lambda_{t+1} \quad (2) \]

\[ \frac{\partial \mathcal{L}}{\partial k_{t+1}} = -\lambda_t + (1 + r) \lambda_{t+1} = 0 \quad \Rightarrow \quad \frac{\lambda_t}{\lambda_{t+1}} = 1 + r \quad (3) \]

Combining (1) and (2), \( \left( \frac{c_{t+1}}{c_t} \right)^{1/\sigma} = \frac{\lambda_t}{\lambda_{t+1}} \delta \). Then \( \frac{c_{t+1}}{c_t} = \left( \frac{\lambda_t}{\lambda_{t+1}} \delta \right)^\sigma \) Substituting from (3)

\[ \frac{c_2}{c_1} = ((1 + r) \delta)^\sigma \quad (4) \]
\[
\frac{c_2}{c_1} = ((1 + r)\delta)^\sigma 
\]  \hspace{1cm} (4)

We will assume that \((1 + r)\delta > 1\) so consumption increases with age.

It will prove convenient to define the parameter \(\alpha\) to satisfy the following equation.

\[(1 + r)\alpha = ((1 + r)\delta)^\sigma\]

Then

\[
\frac{c_{t+1}}{c_t} = (1 + r)\alpha \quad \text{and so} \quad c_t = (1 + r)^{t-1} \alpha^{t-1} c_1 
\]  \hspace{1cm} (5)

Note that as long as \(\sigma < 1, \alpha < 1\). However if \(\sigma\) is sufficiently large \(\alpha\) must exceed 1.

The final step is to solve for first period consumption. To do this we note that the constraints can be written in present value terms as follows.

\[
\frac{k_2}{1 + r} = k_1 + y_1 - c_1 
\]

\[
\frac{k_3}{(1 + r)^2} = \frac{k_2}{1 + r} + \frac{y_2}{1 + r} - \frac{c_2}{1 + r} 
\]

........
Adding these constraints it follows that

\[
\frac{k_{T+1}}{(1 + r)^T} = k_i + \sum_{t=1}^{T} \frac{y_t}{(1 + r)^{t-1}} - \sum_{t=1}^{T} \frac{c_t}{(1 + r)^{t-1}}. \tag{lifetime budget constraint}
\]

Substituting from (5)

\[
\frac{k_{T+1}}{(1 + r)^T} = k_i + PV\{y_i\}_1^T - c_i \sum_{t=1}^{T} \frac{(1 + r)^{t-1} \alpha^{t-1}}{(1 + r)^{t-1}} = k_i + PV\{y_i\}_1^T - c_i \sum_{t=1}^{T} \alpha^{t-1} = k_i + PV\{y_i\}_1^T - c_i \left(\frac{1 - \alpha^T}{1 - \alpha}\right)
\]

Hence

\[
c_i = \left(\frac{1 - \alpha}{1 - \alpha^T}\right) \left(k_i - \frac{k_{T+1}}{(1 + r)^T} + PV\{y_i\}_1^T\right).
\]
Evolution of total wealth

Suppose there is no bequest motive.

The total wealth of the consumer at \( t=1 \) is \( W_1 = k_1 + PV\{y_t\}_{t=1}^T \).

Let \( W_t \) be total wealth in period \( t \). \( W_t = k_t + PV\{y_s\}_{s=1}^T \). The period \( t \) constraint for total wealth is

\[
W_{t+1} = (W_t - c_t)(1 + r)
\]

**Why is this?**

\[
c_1 = \left(\frac{1-\alpha}{1-\alpha^T}\right)W_1
\]

\[
c_2 = \left(\frac{1-\alpha}{1-\alpha^{T-1}}\right)W_2
\]

\[
c_t = \left(\frac{1-\alpha}{1-\alpha^{T+1-t}}\right)W_t
\]
Numerical analysis

Pick any first period consumption. Suppose $W_1 = 100$. We can then compute the entire wealth sequence $\{W_t\}_{t+1}^{T+1}$. The bank won’t allow the borrower to die in debt so $W_{T+1} \geq 0$. The consumer has no bequest motive so does not want to leave assets. Then to maximize lifetime utility $\bar{W}_{T+1} = 0$.

wealth equation $W_{t+1} = (W_t - c_t)(1+r)$

FOC $\frac{c_2}{c_1} = ((1 + r) \delta)\sigma \equiv (1 + r)\alpha$

Computationally this is simple. Pick a $c_1$ and compute $W_{T+1}$ satisfying the FOC and wealth equation. If $W_{T+1} > 0$ pick a larger $c_1$. If $W_{T+1} < 0$ pick a smaller $c_1$. Adjust until the target is approximated sufficiently well.

This is illustrated below. The data is as follows.

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### Choice over time

**Mathematical Foundations**

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![Excel Spreadsheet Image](image-url)
Choice over time
Hitting a target using Solver in EXCEL

The dots indicate the quintiles with 100 periods. Note that for the first three quintiles the solution lies close to the line through the origin indicating that consumption is approximately a constant fraction of total (financial and human) capital.
The next figure shows the solution when there are 200 periods.

Note that the solution is again almost linear but now for the first 4 quintiles. Only in the last 20 periods does the consumer start to consume a much higher fraction of his declining total wealth.
3. One sector macro growth model with strictly concave production function

One representative agent. Economy last for \( T \) periods. Initial capital \( k_1 \) Let \( \{c_t\}_{t=1}^{T} \) be the consumption sequence. Utility is \( U(c) = u(c_1) + \delta u(c_2) + \ldots \) Where \( u \) is concave. We will focus on the special case \( u(c_t) = \ln c_t \). Hence \( u'(c_t) = \frac{1}{c_t} \). Then \( U \) is concave. The depreciation rate is \( \beta \).

output \( y_t = F(k_t) = Ak_t^\gamma \) is either invested or consumed \( c_t + i_t = y_t = Ak_t^\gamma \)

Capital growth equation:

\[
k_{t+1} = k_t - \beta k_t = i_t = Ak_t^\gamma + (1 - \delta)k_t - c_t \quad \text{that is} \quad Ak_t^\gamma + (1 - \beta)k_t - c_t - k_{t+1} \geq 0
\]

\[
\mathcal{L}(c, k, \lambda) = \sum_{t=1}^{T} \delta^{t-1} u(c_t) + \sum_{t=1}^{T} Ak_t^\gamma + (1 - \beta)k_t - c_t - k_{t+1}
\]

\[
= \ldots + \delta^{t-1} u(c_t) + \delta^t u(c_{t+1}) + \lambda_t(Ak_t^\gamma + (1 - \beta)k_t - c_t - k_{t+1}) + \lambda_{t+1}(Ak_{t+1}^\gamma + (1 - \beta)k_{t+1} - c_{t+1} - k_{t+2})
\]
FOC

\[ \frac{\partial L}{\partial c_t} = u'(c_t) - \lambda_t = 0 \Rightarrow \frac{\delta^{t-1}}{c_t} = \lambda_t \]  \hspace{1cm} (1)

\[ \frac{\partial L}{\partial c_{t+1}} = u'(c_{t+1}) - \lambda_{t+1} = 0 \Rightarrow \frac{\delta^t}{c_{t+1}} = \lambda_{t+1} \]  \hspace{1cm} (2)

\[ \frac{\partial L}{\partial k_{t+1}} = \lambda_{t+1} (\gamma A k_{t+1}^{\gamma-1} + 1 - \beta) - \lambda_t = 0 \Rightarrow \frac{\lambda_t}{\lambda_{t+1}} = \gamma A k_{t+1}^{\gamma-1} + 1 - \beta \]  \hspace{1cm} (3)

Combining (1) and (2), \( \frac{c_{t+1}}{c_t} = \delta \frac{\lambda_t}{\lambda_{t+1}} \). Substituting from (3)

\[ \frac{c_{t+1}}{c_t} = \delta (\gamma A k_{t+1}^{\gamma-1} + 1 - \beta) \]  \hspace{1cm} (4)
\[ \frac{c_{t+1}}{c_t} = \delta(\gamma A k_{t+1}^{\gamma -1} + 1 - \beta) \quad \text{FOC} \]

\[ k_{t+1} = k_t - \beta k_t = i_t = A k_t^\gamma + (1 - \delta)k_t - c_t \quad \text{growth equation} \]

**Numerical analysis**

Guess \( \overline{c}_t \) and let the computer figure out \( \overline{k}_{t+1} \).
### Mathematical Foundations

**Choice over time**

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Infinite horizon problems

\[ U = \text{Max} \left\{ \sum_{t=1}^{\infty} u_t(c_t,k_t) \mid k_{t+1} \leq g(c_t,k_t) \right\} \text{ where } u_t(\cdot) \text{ and } g_t(\cdot) \text{ are concave, } t = 1, \ldots \]

Necessary conditions

Suppose that \( \{c_t, k_t \}_{t=1}^{\infty} \) solves this problem. Fix \( k_{T+1} = \bar{k}_{T+1} \). Then \( \{c_t, k_t \}_{t=1}^{T+1} \) must solve

\[ U = \text{Max} \left\{ \sum_{t=1}^{\infty} u_t(c_t,k_t) \mid k_{t+1} \leq g(c_t,k_t), \quad k_{T+1} = \bar{k}_{T+1} \right\} . \]

Thus the FOC for the finite horizon problem must hold for all \( t \).

For any \( c_1 \) the FOC and growth equation map out a sequence \( \{c_t,k_t \}_{t=1}^{\infty} \). Suppose that the sequence \( \{c_t, k_t \}_{t=1}^{\infty} \) satisfies the FOC, growth equation and any other feasibility constraints for all \( t \). Is it optimal?
Concave Optimization problem

\[ \max \left\{ \sum_{t=1}^{\infty} u_t(c_t, k_t) \left| k_{t+1} \leq g(c_t, k_t) \right. \right\} \text{ where } u_t(\cdot) \text{ and } g_t(\cdot) \text{ are concave, } t = 1, \ldots \]

Suppose that the sequence \( \{\bar{c}_t, \bar{k}_t\}_{t=1}^{\infty} \) satisfies the necessary conditions, the growth equation and any other feasibility constraints for all \( t \).

**Malinvaud (sufficient) condition**

If the value of the capital stock approaches zero, that is, \( \{\lambda_t \bar{k}_t\} \to 0 \) then for any other feasible sequence \( \{c_t, k_t\}_{t=1}^{\infty} \) and any \( \varepsilon > 0 \) there exists a \( \hat{T} \) such that for all \( T > \hat{T} \)

\[ \sum_{t=1}^{T} u_t(c_t, k_t) \leq \sum_{t=1}^{T} u_t(\bar{c}_t, \bar{k}_t) + \varepsilon \]

Note that the sums on both sides may not converge to a limit at \( T \) gets large.

The proof follows from the concavity of the \( T \) period Lagrangian and the Kuhn-Tucker conditions for a maximum (see Chapter 6.)