# Constrained Optimization

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AN INTUITIVE APPROACH

\[ \text{Max} \{ f(x) \mid b - g(x) \geq 0, \ x \geq 0 \} . \]

We shall interpret this mathematical problem as the decision problem of a profit maximizing firm. The revenue from the production plan \( x \) is \( f(x) \). The firm can produce any vector of outputs \( x \) which satisfies the non-negativity constraint, \( x \geq 0 \) and a resource constraint

\[ g(x) \leq b. \]

A simple example is a linear constraint. Each unit of \( x_j \) requires \( a_j \) units of resource \( b \). Then

\[ a \cdot x = \sum_{j=1}^{n} a_j x_j \leq b. \]
Suppose that $\bar{x}$ solves the optimization problem. If the firm increases $x_j$, the marginal revenue is,

$$\frac{\partial f}{\partial x_j} (\bar{x})$$. However, the increase in $x_j$ also utilizes additional resources. The extra resource use is $\frac{\partial g}{\partial x_j} (\bar{x})$.

We now introduce a resource price $\lambda$ and ask what price this must be if the profit maximizing decision is to continue to choose $\bar{x}$.

Note that given the resource price $\lambda$ the marginal cost of $x_j$ is $\lambda \frac{\partial g}{\partial x_j} (\bar{x})$.
Suppose that $\bar{x}$ solves the optimization problem. If the firm increases $x_j$, the marginal revenue is,

$$\frac{\partial f}{\partial x_j}(\bar{x}).$$

However, the increase in $x_j$ also utilizes additional resources. The extra resource use is $\frac{\partial g}{\partial x_j}(\bar{x})$.

We now introduce a resource price $\lambda$ and ask what price this must be if the profit maximizing decision is to continue to choose $\bar{x}$.

Note that given the resource price $\lambda$ the marginal cost of $x_j$ is $\lambda \frac{\partial g}{\partial x_j}(\bar{x})$.

If $\bar{x}_j > 0$ marginal revenue must be equal to marginal cost so

$$MR_j - MC_j = \frac{\partial f}{\partial x_j}(\bar{x}) - \lambda \frac{\partial g}{\partial x_j}(\bar{x}) = 0.$$  

If $\bar{x}_j = 0$ the marginal profit must be negative so

$$MR_j - MC_j = \frac{\partial f}{\partial x_j}(\bar{x}) - \lambda \frac{\partial g}{\partial x_j}(\bar{x}) \leq 0.$$  

If there is no market price we follow the same approach. Instead of the market price we introduce a “shadow price” $\lambda \geq 0$ to reflect the opportunity cost of using the additional resources. The FOC are then the conditions above with the “as if” market price if there is no actual market price.
Summarizing our results

\[ \frac{\partial f}{\partial x_j}(\bar{x}) - \lambda \frac{\partial g}{\partial x_j}(\bar{x}) \leq 0, \] with equality if \( \bar{x}_j > 0 \).

Equivalently:

\[ \frac{\partial f}{\partial x_j}(\bar{x}) - \lambda \frac{\partial g}{\partial x_j}(\bar{x}) \leq 0, \quad \text{and} \quad \bar{x}_j \left( \frac{\partial f}{\partial x_j}(\bar{x}) - \lambda \frac{\partial g}{\partial x_j}(\bar{x}) \right) = 0. \]

The second condition is called a **complementary slackness condition** as it can only hold if one but not both of the conditions (i) \( MR_j - MC_j = \frac{\partial f}{\partial x_j}(\bar{x}) - \lambda \frac{\partial g}{\partial x_j}(\bar{x}) \leq 0 \) and (ii) \( \bar{x}_j \geq 0 \) is not binding (i.e. “slack”).
Since \( \bar{x} \) must be feasible \( b - g(\bar{x}) \geq 0 \). Moreover, we have defined \( \lambda \geq 0 \) to be the opportunity cost of additional resource use. Then if not all the resource is used, \( \lambda \) must be zero. Summarizing,

\[
b - g(\bar{x}) \geq 0, \text{ with equality if } \lambda > 0.
\]

Equivalently:

\[
b - g(\bar{x}) \geq 0 \text{ and } \lambda(b - g(\bar{x})) = 0.
\]

Once again at most one of the inequalities \( b - g(x) \geq 0 \) and \( \lambda \geq 0 \) can be “slack”. So this is a second complementary slackness condition.

**Generalization to multiple constraints**

Suppose that there are \( m \) constraints and we write these as follows \( h_i(x) \geq 0, \ i = 1,\ldots,m \) Arguing in the same way we introduce a shadow price \( \lambda_i \) for the \( i \)th constraint.

**FOC:** \[
MR_j - MC_j = \frac{\partial f}{\partial x_j} - \sum_{i=1}^{m} \frac{\partial g_i}{\partial x_j} \leq 0 \quad \text{with equality if } \bar{x}_j > 0 \quad j = 1,\ldots,n
\]
There is a convenient way to remember these conditions. Write the $i$-th constraint in the form

$$ h_i(x) \geq 0, \quad i = 1,\ldots,m. $$

In vector notation $h(x) \geq 0$.

(Thus in our example we write the constraint as $h_i(x) = b - g(x) \geq 0$.)

Then introduce a vector of “Lagrange multipliers” or shadow prices $\lambda$ and define the Lagrangian

$$ \mathcal{L}(x, \lambda) = f(x) + \lambda \cdot h(x) $$

The first order conditions are then all restrictions on the partial derivatives of $\mathcal{L}(x, \lambda)$.

(i) \[ \frac{\partial \mathcal{L}}{\partial x_j} = \frac{\partial f}{\partial x_i} + \lambda \cdot \frac{\partial h}{\partial x_j} \leq 0, \text{ with equality if } \lambda_j > 0, \quad j = 1,\ldots,n. \]

(ii) \[ \frac{\partial \mathcal{L}}{\partial \lambda_i} = h_i(x) \geq 0, \text{ with equality if } \lambda_i > 0, \quad i = 1,\ldots,m. \]

Exercise: Solve the following problem. $\text{Max}\{U(x) = \ln x_1 + \ln(x_2 + 2x_3) \mid p_1x_1 + p_2x_2 + p_3x_3 \leq 60\}$

(i) if $p = (1, 2, 6)$ (ii) $p = (1, 2, 2)$ (iii) $p = (1, 2, 4)$
Restatement of FOC

(i) \[ \frac{\partial \mathcal{L}}{\partial x_i} = \frac{\partial f}{\partial x_i} + \lambda \cdot \frac{\partial h}{\partial x_i} \leq 0 \quad \text{and} \quad \bar{x}_j \frac{\partial \mathcal{L}}{\partial x_j} = 0 \quad , \quad j = 1,\ldots,n. \]

(ii) \[ \frac{\partial \mathcal{L}}{\partial \lambda_i} = h_i(\bar{x}) \geq 0 \quad \text{and} \quad \lambda_i \frac{\partial \mathcal{L}}{\partial \lambda_i} = 0 \quad , \quad i = 1,\ldots,m. \]

In vector notation

(i) \[ \frac{\partial \mathcal{L}}{\partial x} = \frac{\partial f}{\partial x} + \lambda \cdot \frac{\partial h}{\partial x} \leq 0 \quad \text{and} \quad \bar{x} \cdot \frac{\partial \mathcal{L}}{\partial x} = 0 \]

(ii) \[ \frac{\partial \mathcal{L}}{\partial \lambda} = h(\bar{x}) \geq 0 \quad \text{and} \quad \lambda \cdot \frac{\partial \mathcal{L}}{\partial \lambda} = 0 \]

BUT…..

It turns out that this intuitive argument is not quite correct. It breaks down in some exceptional cases. To avoid these we some very mild impose mild restrictions on the constraints. These are called “constraint qualifications”.
**Constraint Qualifications**

Consider the maximization problem  \( \max \{ f(x) \mid x \in X \} \) where  \( X = \{ x \mid x \geq 0, \ h_i(x) \geq 0, \ i = 1, \ldots, m \} \).

The constraint qualifications holds at  \( \bar{x} \in X \) if

(i) for each constraint that is binding at  \( \bar{x} \) the associated gradient vector  \( \frac{\partial h_i}{\partial x} (\bar{x}) \neq 0 \).

(ii)  \( \bar{X} \), the set of non-negative vectors satisfying the linearized binding constraints has a non-empty interior.

Note: The linear function  \( h_i^L (x) = h_i(\bar{x}) + \frac{\partial h_i}{\partial x} (\bar{x}) \cdot (x - \bar{x}) \) has the same value and gradient as the function  \( h_i(x) \) at  \( \bar{x} \). Thus the linear approximation at  \( \bar{x} \) of the ith constraint  \( h_i(x) \geq 0 \) is

\[
h_i(\bar{x}) + \frac{\partial h_i}{\partial x} (\bar{x}) \cdot (x - \bar{x}) \geq 0.
\]

Since  \( h_i(\bar{x}) = 0 \) the linearized constraint is  \( \frac{\partial h_i}{\partial x} (\bar{x}) \cdot (x - \bar{x}) \geq 0 \).
**Remark: Satisfying the second constraint qualification**

At the end of these slides is a proof of the following proposition.

**Proposition:** Suppose that we re-label the constraints so that it is the first $I$ that are binding at $x^0$, that is $h_i(x^0) = 0, \ i = 1,\ldots,I$. Suppose also that there are $J$ variables for which the non-negativity constraint is binding. Then consider the $I$ linearized binding constraints and $J$ binding non-negativity constraints. This is a system of $I+J$ linear constraints $A(x - x^0) = 0$. If the rows of $A$ are linearly independent then the second constraint qualification holds at $x^0$.

As a practical matter, this result is almost never employed in economic analysis. The reason is that economists typically make assumptions that imply that the feasible set $X$ is convex and has a non-empty interior. As we shall see, when these assumptions hold, checking the first constraint qualification is enough.
Example 1: Constraint qualification holds

\[ Max \{ f(x) = \ln x_1 x_2 \mid x \geq 0, \ h(x) = 2 - x_1 - x_2 \geq 0 \} . \]

As is readily confirmed, the maximizing value of \( x \) is \( \bar{x} = (1,1) \).

The feasible set and contour set for \( f \) through \( \bar{x} = (1,1) \) are depicted.

The Lagrangian is \( \mathcal{L}(x, \lambda) = \ln x_1 + \ln x_2 + \lambda (2 - x_1 - x_2) \).

The first order conditions are therefore

(i) \( \frac{\partial \mathcal{L}}{\partial x_j} = \frac{1}{x_j} - \lambda \leq 0, \text{ with equality if } x_j > 0, \ j=1,2 \)

(ii) \( \frac{\partial \mathcal{L}}{\partial \lambda} = 2 - x_1 - x_2 \geq 0, \text{ with equality if } \lambda > 0. \)

As is readily checked, the necessary conditions are all satisfied at \( (\bar{x}, \bar{\lambda}) = (1,1,1) \).
Example 2: Constraint qualification does not hold

\[
\text{Max}\{ f(x) = \ln x_1 + \ln x_2 \mid x \geq 0, \ h(x) = (2 - x_1 - x_2)^3 \geq 0 \}.
\]

Since the feasible set and maximand are exactly the same as in example 1, the solution is again \( \bar{x} = (1, 1) \).

The Lagrangian is

\[
\mathcal{L}(x, \lambda) = \ln x_1 + \ln x_2 + \lambda (2 - x_1 - x_2)^3.
\]

Differentiating by \( x_j \),

\[
\frac{\partial \mathcal{L}}{\partial x_j} = \frac{1}{x_j} - 3\lambda (2 - x_1 - x_2)^2 = 1 \text{ at } \bar{x} = (1, 1).
\]
Thus the first order condition does not hold at the maximum. Our intuitive argument breaks down because \( \frac{\partial h}{\partial x} \), the gradient of the constraint function, is zero at the maximum. Thus the partial derivatives of \( h \) no longer reflect the opportunity cost of the scarce resource. More formally, at the maximum \( \bar{x} \), the linear approximation of the constraint is

\[
h_i^L(x) = h_i(\bar{x}) + \frac{\partial h_i}{\partial x}(\bar{x}) \cdot (x - \bar{x}).
\]

Then, as long as the constraint is binding (\( h_i(\bar{x}) = 0 \)) and the gradient vector is not zero, the linearized constraint is

\[
\frac{\partial h_i}{\partial x}(\bar{x}) \cdot (x - \bar{x}) \geq 0, \quad i = 1, \ldots, m.
\]
There is a second (though unlikely) situation in which the linear approximations fail.

Consider the following problem.

\[ \max_{x} \{ f(x) = 12x_1 + x_2 \mid h(x) = (2 - x_1)^3 - x_2 \geq 0, x \geq 0 \} . \]

The feasible set is the shaded region in the figure.

Also depicted is the contour set for \( f \) through \( \bar{x} = (2, 0) \).

From the figure it is clear that \( f \) takes on its maximum at \( \bar{x} = (2, 0) \).

However, \[ \frac{\partial \mathcal{L}}{\partial x_1}(\bar{x}) = 12 - 3\lambda(\bar{x}_1 - 2)^2 = 12 . \]

Thus again the first order conditions do not hold at the maximum.
This time the problem occurs because the feasible set, after taking a linear approximation of the constraint function, looks nothing like the original feasible set.

At $\bar{x}$, the gradient vector $\frac{\partial h}{\partial x}(x) = (0, -1)$.

Thus the linear approximation of the constraint $h(x) \geq 0$ through $\bar{x}$ is

$$\frac{\partial h}{\partial x}(\bar{x}) \cdot (x - \bar{x}) = \frac{\partial h}{\partial x_1}(\bar{x})(x_1 - 2) + \frac{\partial h}{\partial x_2}(\bar{x})x_2 = -x_2 \geq 0.$$ 

Since $x$ must be non-negative, the only feasible value of $x_2$ is $x_2 = 0$. In Figure 1.2-2b the linearized feasible set is therefore the horizontal axis. Then the solution to the linearized problem is not the solution to the original problem.
As long as the constraint qualifications hold, the intuitively derived conditions are indeed necessary conditions. This is summarized below.

**Proposition: First Order Conditions for a Constrained Maximum**

Suppose $\bar{x}$ solves $\underset{x}{\text{Max}}\{f(x) \mid x \in X\}$ where $X = \{x \mid x \geq 0, \ h(x) \geq 0\}$.

If the constraint qualifications hold at $\bar{x}$ then there exists a vector of shadow prices $\lambda \geq 0$ such that

$$\frac{\partial L}{\partial x_j}(\bar{x}, \lambda) \leq 0, \ j = 1, \ldots, n \text{ with equality if } \bar{x}_j > 0$$

and

$$\frac{\partial L}{\partial \lambda}(\bar{x}, \lambda) \geq 0, \ i = 1, \ldots, m \text{ with equality if } \lambda_i > 0$$

Kuhn-Tucker conditions

____________________________

This result is usually attributed to Kuhn and Tucker. However the first formal statement of the conditions is by Karush (1939).
Special case: Concave optimization problem

For the optimization problem \( \max_x \{ f(x) \mid x \in X \} \) where \( X = \{ x \mid x \geq 0, \ h(x) \geq 0 \} \)

suppose that \( f, h_1, \ldots, h_m \) are all concave functions. Then if we can find some \((\bar{x}, \lambda)\) satisfying the FOC we don’t have to worry about the constraint qualifications.

Example:

\( \max_x \{ u(x) \mid p \cdot x \leq I, x \in \mathbb{R}^n_+ \} \) where \( u \) is a concave function

\( h(x) = I - \sum_{j=1}^{n} p_j x_j \)

\(-p_j x_j\) is a concave function and the sum of concave functions is concave therefore \( h(x) \) is concave
Proposition: If conditions (i) and (ii) hold at \((\bar{x}, \lambda)\) then \(\bar{x}\) solves \(\max_{x} \{f(x) \mid x \in X\}\) where

\[
X = \{x \mid x \geq 0, \quad h(x) \geq 0\}
\]

(i) \[\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial f}{\partial x} + \lambda \cdot \frac{\partial h}{\partial x} \leq 0 \quad \text{and} \quad \bar{x} \cdot \frac{\partial \mathcal{L}}{\partial x} = 0\]

(ii) \[\frac{\partial \mathcal{L}}{\partial \lambda} = h(\bar{x}) \geq 0 \quad \text{and} \quad \lambda \cdot \frac{\partial \mathcal{L}}{\partial \lambda} = \lambda \cdot h(\bar{x}) = 0, \quad i = 1, \ldots, m.\]

Proof:

The sum of concave functions is concave so \(\mathcal{L}(x, \lambda) = f(x) + \lambda \cdot h(x)\) is a concave function of \(x\).

Hence \(\mathcal{L}(x, \lambda) \leq \mathcal{L}(\bar{x}, \lambda) + \frac{\partial \mathcal{L}}{\partial x}(\bar{x}, \lambda) \cdot (x - \bar{x})\).

As you can check from the FOC, the second term on the right hand side is negative.

Also, appealing to (ii) \(\mathcal{L}(\bar{x}, \lambda) = f(\bar{x})\).

Therefore \(\mathcal{L}(x, \lambda) = f(x) + \lambda \cdot h(x) \leq f(\bar{x})\). But \(\lambda \cdot h(x) \geq 0\) for any feasible \(x\).

Therefore \(f(x) \leq f(\bar{x})\).
**Proposition: Sufficient Conditions for a Maximum**

Suppose that \( f \) and \( h_i, i = 1, \ldots, m \) are quasi-concave and the feasible set \( X = \{ x \mid x \geq 0, h(x) \geq 0 \} \) has a non-empty interior.

If the Kuhn-Tucker conditions hold at \( \bar{x} \), \( \frac{\partial f}{\partial x}(\bar{x}) \neq 0 \)

and for each binding constraint, \( \frac{\partial h_i}{\partial x}(\bar{x}) \neq 0 \),

then \( \bar{x} \) solves \( \max_{x} \{ f(x) \mid x \geq 0, h_i(x) \geq 0, \ i = 1, \ldots, m \} \).

**Intuition:**

Under these assumptions the feasible set is convex and hence the linearized feasible set \( \bar{X} \) contains the original feasible set \( X \). Then it is sufficient to show that \( X \) has a non-empty interior.
**Application: Joint Costs**

An electricity company has an interest cost $c_o = 20$ per day for each unit of turbine capacity. For simplicity we define a unit of capacity as megawatt. It faces day-time and night-time demand price functions as given below.

The operating cost of running a each unit of turbine capacity is 10 in the day time and 10 at night.

\[ p_1 = 200 - q_1, \quad p_2 = 100 - q_2 \]

Formulating the problem

Solving the problem

Understanding the solution (developing the economic insights.)
Try a simple approach.

Each unit of turbine capacity costs 10+10 to run each day plus has an interest cost of 20 so $MC=40$

With $q$ units sold the sum of the demand prices is $300 - 2q$ so revenue is $TR = q(300 - 2q)$ hence $MR = 300 - 4q$. Equate MR and MC. $q^* = 65$. 
Try a simple approach.

Each unit of turbine capacity costs 10+10 to run each day plus has an interest cost of 20 so MC=40

With $q$ units sold the sum of the demand prices is $300 - 2q$ so revenue is $TR = q(300 - 2q)$ hence $MR = 300 - 4q$. Equate $MR$ and $MC$. $q^* = 65$.

Lets take a look on the margin in the day and night time

$TR_1 = q_1(200 - q_1)$ then $MR_1 = 200 - 2q_1$.

$TR_2 = q_2(200 - q_2)$ then $MR_2 = 100 - 2q_2$.

$MR_2 < 0$ !!!!!

Now what?
Key is to realize that there are really three variables. Production in each of the two periods \(q_1\) and \(q_2\) and the plant capacity \(q_0\).

\[
C(q) = 20q_0 + 10q_1 + 10q_2
\]

\[
R(q) = R_1(q_1) + R_2(q_2) = (200 - q_1)q_1 + (100 - q_2)q_2
\]

Constraints

\[
h_1(q) = q_0 - q_1 \geq 0 \quad h_2(q) = q_0 - q_2 \geq 0.
\]
Key is to realize that there are really three variables. Production in each of the two periods $q_1$ and $q_2$ and the plant capacity $q_0$.

$C(q) = c \cdot q$ where $c = (c_0, c_1, c_2) = (20, 10, 10)$

$R(q) = R_1(q_1) + R_2(q_2) = (200 - q_1)q_1 + (100 - q_2)q_2$

Constraints: $h_1(q) = q_0 - q_1 \geq 0 \quad h_2(q) = q_0 - q_2 \geq 0$.

$L = R_1(q_1) + R_2(q_2) - c_0q_0 - c_1q_1 - c_2q_2 + \lambda_1(q_0 - q_1) + \lambda_2(q_0 - q_2)$

Kuhn-Tucker conditions

\[
\frac{\partial L}{\partial q_j} = MR_j(q_j) - c_j - \lambda_j \leq 0, \text{ with equality if } \bar{q}_j > 0, j=1,2
\]

\[
\frac{\partial L}{\partial q_0} = -c_0 + \lambda_1 + \lambda_2 \leq 0, \text{ with equality if } \bar{q}_0 > 0
\]

\[
\frac{\partial L}{\partial \lambda_i} = q_0 - q_i \geq 0, \text{ with equality if } \lambda_i > 0, i=1,2
\]
Kuhn-Tucker conditions

\[ \frac{\partial \mathcal{L}}{\partial q_1} = 200 - 2q_1 - 10 - \lambda_1 \leq 0, \text{ with equality if } \bar{q}_1 > 0 \]

\[ \frac{\partial \mathcal{L}}{\partial q_2} = 100 - 2q_2 - 10 - \lambda_2 \leq 0, \text{ with equality if } \bar{q}_1 > 0 \]

\[ \frac{\partial \mathcal{L}}{\partial q_0} = -20 + \lambda_1 + \lambda_2 \leq 0, \text{ with equality if } \bar{q}_0 > 0 \]

\[ \frac{\partial \mathcal{L}}{\partial \lambda_1} = q_0 - q_1 \geq 0, \text{ with equality if } \lambda_1 > 0. \]

\[ \frac{\partial \mathcal{L}}{\partial \lambda_2} = q_0 - q_2 \geq 0, \text{ with equality if } \lambda_2 > 0. \]

Solve by trial and error.

(i) Suppose \( \bar{q}_0 > 0 \) and both shadow prices are positive. Then \( q_0 = q_1 = q_2 \).

(ii) Solve and you will find that one of the shadow prices is negative. But this is impossible.

(iii) Inspired guess. That shadow price must be zero. Then the positive shadow price must be 20.
Technical Remark: Satisfying the second constraint qualification

**Proposition:** Suppose that we re-label the constraints so that it is the first $I$ that are binding at $x^0$, that is $h_i(x^0) = 0$, $i = 1, ..., I$. Suppose also that there are $J$ variables for which the non-negativity constraint is binding. Then consider the $I$ linearized binding constraints and $J$ binding non-negativity constraints. This is a system of $K = I + J$ linear constraints $A(x - x^0) = 0$ If the rows of $A$ are linearly independent then the second constraint qualification holds at $x^0$.

**Proof:** Suppose that at $x^0$ there are $I$ binding constraints. We re-label these so that

$h_i(x^0) = 0$, $i = 1, ..., I$ and $h_i(x^0) > 0$, $i > I$.

The linearized feasible set is therefore $\bar{X} = \{ x \mid x \geq 0, \frac{\partial h_i}{\partial x}(x^0) \cdot (x - x^0) \geq 0, \ i = 1, ..., I \}$.

Suppose that there are $J$ variables for which $x_j^0 = 0$. We re-label these variables so that $x_j^0 = 0$, $j = I + 1, ..., I + J$.

Thus there are $K = I + J$ binding constraints in the linearized feasible set.

Suppose we choose

$x_j = x_j^0$, $j > K$.  \hspace{1cm} (1)
Then the $K$ binding constraints can be re-written as a system of $K$ linear equations in $K$ unknowns.

$$
\begin{bmatrix}
\frac{\partial h_1(x^0)}{\partial x_1} & \frac{\partial h_1(x^0)}{\partial x_I} & \frac{\partial h_1(x^0)}{\partial x_{I+1}} & \frac{\partial h_1(x^0)}{\partial x_K} \\
\vdots & \vdots & \vdots & \vdots \\
\frac{\partial h_I(x^0)}{\partial x_1} & \frac{\partial h_I(x^0)}{\partial x_I} & \frac{\partial h_I(x^0)}{\partial x_{I+1}} & \frac{\partial h_I(x^0)}{\partial x_K} \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & . \\
0 & 0 & 0 & . \\
\end{bmatrix}
\begin{bmatrix}
x_1 - x_1^0 \\
\vdots \\
x_I - x_I^0 \\
x_{I+1} - x_{I+1}^0 \\
x_K - x_K^0 \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
\vdots \\
0 \\
0 \\
0 \\
\end{bmatrix}.
$$

(2)

Choose $b_k \geq 0$, $k = 1, \ldots, K$ and consider the following linear equation system.

$$
\begin{bmatrix}
\frac{\partial h_1(x^0)}{\partial x_1} & \frac{\partial h_1(x^0)}{\partial x_I} & \frac{\partial h_1(x^0)}{\partial x_{I+1}} & \frac{\partial h_1(x^0)}{\partial x_n} \\
\vdots & \vdots & \vdots & \vdots \\
\frac{\partial h_I(x^0)}{\partial x_1} & \frac{\partial h_I(x^0)}{\partial x_I} & \frac{\partial h_I(x^0)}{\partial x_{I+1}} & \frac{\partial h_I(x^0)}{\partial x_I} \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & . \\
0 & 0 & 0 & . \\
\end{bmatrix}
\begin{bmatrix}
x_1 - x_1^0 \\
\vdots \\
x_I - x_I^0 \\
x_{I+1} - x_{I+1}^0 \\
x_K - x_K^0 \\
\end{bmatrix}
= \begin{bmatrix}
b_1 \\
\vdots \\
b_I \\
b_{I+1} \\
b_K \\
\end{bmatrix}.
$$

(3)
If the columns (or rows) of the matrix are independent, then the matrix is invertible. Then for each \( b \) there is a unique solution \( x_k(b), \ k = 1,\ldots, K \). For \( k > K \) define \( x_k(b) = x_0^k \).

Consider any \( b >> 0 \). From (3) it follows that

\[
\sum_{j=1}^{n} \frac{\partial h_i}{\partial x_j} (x^0)(x_j(b) - x^0) = b_i > 0, \ i = 1,\ldots, I \}
\]

and

\[
x_j(b) = b_j, \ j = I + 1,\ldots, K.
\]

Remember that \( x_k(b) = x_0^k > 0, \ k > K \). Also \( x_k(0) > 0, \ k = 1,\ldots, I \).

Then for \( b >> 0 \) and sufficiently small \( x_k(b) > 0, \ k = 1,\ldots, I \). Thus for \( b >> 0 \) and sufficiently small

\[
\sum_{j=1}^{n} \frac{\partial h_i}{\partial x_j} (x^0)(x_j(b) - x^0) > 0, \ i = 1,\ldots, I \} \text{ and } x(b) >> 0.
\]

Thus \( \bar{X} \) has a non-empty interior.

Q.E.D.
**Exercises**

1. **Utility maximization**

   \[ u(x) = 12x_1 - x_1^2 + 4x_2 - x_2^2 \]  
   The price vector is \( p = (1,1) \).

   Solve for the utility maximizing consumption vector if (i) \( I = 6 \) (ii) \( I = 10 \).

2. **Utility maximization**

   \[ u(x) = 12 \ln x_1 + 4x_2 - x_2^2 \]  
   The price vector is \( p = (1,1) \).

   (a) Solve for the utility maximizing consumption vector if \( I = 7 \).

   (b) For what income levels (if any) is consumption of commodity 2 equal to zero?

3. **Cost minimization**

   The maximum output that a firm can produce with input vector \( z = (z_1, z_2) \) is \( q = (z_1 + 2z_2^{1/2})^2 \). (This mapping is called a production function.)

   (a) Confirm that the constraint qualifications hold for any feasible input vector \( \bar{x} > 0 \)

   (b) If the input price vector is \( r \), solve for the cost minimizing inputs and hence the minimum cost under the assumption that \( q > (r_1 / r_2)^2 \).

   (c) Show that marginal cost is an increasing function for \( q > (r_1 / r_2)^2 \).
(d) Show also that an increase in one of the two input prices has no effect on marginal cost. Do you find this puzzling?

(e) Solve also for minimized cost if \( q < \left( \frac{r_1}{r_2} \right)^2 \).

(f) Is marginal cost \( MC(q) \) continuous on \( \mathbb{R}_+ \)?

4. Peak and off-peak pricing

An electric power company divided the day into three equal periods. The demand prices in the three periods are as follows.

\[ p_1 = a_1 - \frac{1}{2}q_1, \quad p_2 = 220 - q_2, \quad p_3 = 200 - q_3 \]

The unit cost of producing electricity in each period is 10. If the capacity of the firm is \( q \) (so that the maximum output in each period is \( q \)) the interest cost per day is 40\( q \). That is, the unit cost of capacity is 40.

(a) If \( a_1 = 140 \) solve for the profit maximizing outputs and prices in each period.

(b) Solve again if \( a_1 = 160 \).

(c) Returning to the first case, show that the total (i.e. gross) benefit of the production plan \((q_1, q_2, q_3)\) is

\[ B = 140q_1 - \frac{1}{4}q_1^2 + 220q_2 - \frac{1}{2}q_2^2 + 200q_3 - \frac{1}{2}q_3^2 \]

(d) Show that social surplus \((SS = B-C)\) is maximized with a plan twice as large as the profit maximizing plan.
**Answer to Exercise 3**

The production function $q = F(z)$ is the maximum output that can be produced using input vector $z$. Thus the set $Z = \{ z \mid F(z) - q \geq 0 \}$ is the set of feasible input vectors.

As depicted, cost is minimized at $z \gg 0$. If you check the slopes at A and B you will see why one of these points will be cost minimizing for some range of input price ratios while the other is never cost minimizing.

Note that if $q = 0$ the cost minimizing input vector is $z = 0$.

Henceforth we consider $q > 0$.

Then for feasibility, $z > 0$.

Define $h(z) = (z_1 + 2z_2^{1/2})^2 - q$. Then

$$\frac{\partial h}{\partial z} = (2(z_1 + 2z_2^{1/2}), 2(z_1 + 2z_2^{1/2})z_2^{-1/2}) .$$
Constraint Qualifications

Note that \( \frac{\partial h}{\partial z_1}(z) > 0 \) if \( z > 0 \). Then the first CQ is satisfied.

Note next that \( z_1 + 2z_2^{1/2} \) is concave since it is the sum of two concave functions. Since \( y^2 \) is strictly increasing on \( \mathbb{R}_+ \) it follows that \( h(z) = (z_1 + 2z_2^{1/2})^2 - q \) is strictly quasi-concave. Then we only need to check that the feasible set \( Z = \{ z \mid F(z) - q \geq 0 \} \) has a non-empty interior. Note that if \( z = (q^{1/2}, q) \) then \( h(z) = (z_1 + 2z_2^{1/2})^2 - q = 8q > 0 \). Thus the interior of \( Z \) is indeed non-empty.

Minimization problem

\[
\min_{z \geq 0} \{ r_1z_1 + r_2z_2 \mid (z_1 + 2z_2^{1/2})^2 - q \geq 0 \}
\]

Note that if the constraint is not binding, then cost can be reduced by reducing \( z \). Thus at \( \bar{z} \) the constraint must be binding.

Maximization problem

\[
\max_{z \geq 0} \{ -r_1z_1 - r_2z_2 \mid (z_1 + 2z_2^{1/2})^2 - q \geq 0 \}
\]
The Lagrangian for this optimization problem is

\[ \mathcal{L} = -r_1 z_1 - r_2 z_2 + \lambda ((z_1 + 2z_2^{1/2})^2 - q) . \]

Kuhn-Tucker conditions

\[ \frac{\partial \mathcal{L}}{\partial z_1} = -r_1 + 2\lambda (z_1 + 2z_2^{1/2}) \leq 0 \text{ with equality if } \bar{z}_1 > 0 . \quad (1) \]

\[ \frac{\partial \mathcal{L}}{\partial z_2} = -r_2 + 2\lambda (z_1 + 2z_2^{1/2})z_2^{-1/2} \leq 0 \text{ with equality if } \bar{z}_2 > 0 . \quad (2) \]

\[ \frac{\partial \mathcal{L}}{\partial \lambda} = (z_1 + 2z_2^{1/2})^2 - q \geq 0 \text{ with equality if } \lambda > 0 . \quad (3) \]

Since \( q > 0 \), \( \bar{z} > 0 \).

If \( \bar{z}_1 > 0 \) (1) is an equality and so \( \lambda > 0 \).

If \( \bar{z}_2 > 0 \) (2) is an equality and so again \( \lambda > 0 \).
We can rewrite (1) and (2) as follows:

\[ 2\lambda (z_1 + 2z_2^{1/2}) \leq r_1 \quad \text{with equality if } z_1 > 0. \quad (1) \]

\[ 2\lambda (z_1 + 2z_2^{1/2}) - r_2 z_2^{1/2} \leq 0 \quad \text{with equality if } z_2 > 0. \quad (2) \]

Case (i)

Try \( z >> 0 \). Then both (1) and (2) are equalities and so \( r_1 = r_2 z_2^{1/2} \).

Then \( z_2^{1/2} = \frac{r_1}{r_2} \) and so \( z_2 = (\frac{r_1}{r_2})^2 \).

From (3) \( (z_1 + 2z_2^{1/2})^2 = q \). Therefore \( z_1 = q^{1/2} - 2(\frac{r_1}{r_2}) \).

Note that since we assumed that \( z_1 > 0 \), \( q^{1/2} - 2(\frac{r_1}{r_2}) > 0 \), that is \( q > 4(\frac{r_1}{r_2})^2 \).

Total cost is

\[ C(q) = r \cdot z = r_1 q^{1/2} - \frac{r_1^2}{r_2} \]
Case (ii)

Try \( z_1 = 0 \). Then from (3) \( 4z_2 = q \).

Then

\[
C(q) = r_2 z_2 = \frac{r_2 q}{4} \quad \text{for} \quad q \leq 4\left(\frac{r_1}{r_2}\right)^2.
\]

Note: For a complete solution you should check that equality (1) and inequality (2) are both satisfied.

Marginal cost

\[
MC = \frac{dC}{dq} = \frac{r_2}{4}, \quad q \leq 4\left(\frac{r_1}{r_2}\right)^2
\]

\[
MC = \frac{dC}{dq} = \frac{1}{2} \frac{r_1}{q_2^{1/2}}, \quad q > 4\left(\frac{r_1}{r_2}\right)^2
\]

If you check you will find that MC is continuous.