Convex set
Convex function
Strictly Convex function
Concave and strictly concavely concave functions
Quasi-concave function
Supporting hyperplane
**Convex combination**

$x^\lambda$ is a convex combination of $x^0, x^1 \in \mathbb{R}^n$, if for some $\lambda \in (0,1)$

$$x^\lambda = (1 - \lambda)x^0 + \lambda x^1$$

**Convex Set**

$S \subset \mathbb{R}^n$ is convex if, for each $x^0, x^1 \in S$ every convex combination $x^\lambda = (1 - \lambda)x^0 + \lambda x^1 \in S$

**Strictly convex set**

The function $f$ is strictly convex if, for each $x^0, x^1 \in S$ every convex combination $x^\lambda$ lies in the interior of $S$ ($x^\lambda \in \text{int } S$)
We consider functions $f$ defined over a convex set $S \subset \mathbb{R}^n$

**C1:** $f$ is a convex function on $S$ if, for any $x^0, x^1 \in S$ and convex combination $x^\lambda$, $0 < \lambda < 1$,

$$f(x^\lambda) \leq (1 - \lambda)f(x^0) + \lambda f(x^1)$$

**SC1:** $f$ is a strictly convex function on $S$ if, for any $x^0, x^1 \in S$ and convex combination $x^\lambda$, $0 < \lambda < 1$,

$$f(x^\lambda) < (1 - \lambda)f(x^0) + \lambda f(x^1)$$
What is the relationship between a convex set and a convex function?

**Definition: Graph of a function** \( f : S \subset \mathbb{R}^n \to \mathbb{R} \)

\[ G = \{(x, y) \in \mathbb{R}^{n+1} \mid x \in S, \ y = f(x)\} \]

A function is convex if and only if the set

\[ A = \{(x, y) \mid y \geq f(x), \ x \in S\} \text{ is a convex set} \]

**Exercise:** Prove that if \( A = \{(x, y) \mid y \geq f(x), \ x \in S\} \) is a convex set then \( f \) is a convex function.

**HINT:** Suppose \((x^0, y^0)\) and \((x^1, y^1)\) are both in \( A \). Then \( y^0 \geq f(x^0) \) and \( y^0 \geq f(x^0) \).

Then appeal to the convexity of \( f \).

**Exercise:** Prove that if \( f \) is convex then \( A = \{(x, y) \mid y \geq f(x), \ x \in S\} \) is a convex set.

**HINT:** Consider \((x^0, y^0) = (x^0, f(x^0))\) and \((x^1, y^1) = (x^1, f(x^1))\). By C2 the convex combination
\[ (x^2, y^2) = (x^2, (1 - \lambda)f(x^0) + \lambda f(x^1)) \in A \]
Alternative definition of a convex function

**Key idea:**

For and $x^0, x^1 \in S$ we can write the linear combinations where the weights sum to 1, as follows:

$$x(\theta) = (1 - \theta)x^0 + \theta x^1$$

We the consider the properties of the mapping $g(\theta) \equiv f(x(\theta))$. 

![Diagram of convex function with points labeled $x^0 = x(0)$ and $x^1 = x(1)$ on a line passing through these points.]
\[ g(\theta) \equiv f(x(\theta)) \]

\[ x(\theta) = (1 - \theta)x^0 + \theta x^1 = x^0 + \theta(x^1 - x^0) \]
C2: For any $x^0, x^1 \in S$ define $x(\theta) = (1 - \theta)x^0 + \theta x^1$ and $g(\theta) = f(x(\theta))$. Then
\[ x(\theta^0), x(\theta^1) \in S \Rightarrow g((1 - \lambda)\theta^0 + \lambda \theta^1) \geq (1 - \lambda)g(\theta^0) + \lambda g(\theta^1) \]

CS2: For any $x^0, x^1 \in S$ define $x(\theta) = (1 - \theta)x^0 + \theta x^1$ and $g(\theta) = f(x(\theta))$. Then
\[ x(\theta^0), x(\theta^1) \in S \Rightarrow g((1 - \lambda)\theta^0 + \lambda \theta^1) > (1 - \lambda)g(\theta^0) + \lambda g(\theta^1) \]
**Proposition:** $C1 \iff C2$ and $CS1 \iff CS2$

**Proof:**

$C2 \Rightarrow C1$:

If $C2$ holds, $x(0) = x^0$ and $x(1) = x^1$ and so $g(\lambda) \geq (1 - \lambda)g(0) + \lambda g(1)$, that is $f(x^\lambda) \equiv f(x(\lambda)) \geq (1 - \lambda)f(x^0) + \lambda f(x^1)$.

$C1 \Rightarrow C2$:

$x(\theta)$ is linear therefore $x(\theta^\lambda) \equiv x((1 - \lambda)\theta^0 + \lambda \theta^1) = (1 - \lambda)x(\theta^0) + \lambda x(\theta^1)$.

Since $f$ is convex, $g(\theta^\lambda) \equiv f(x(\theta^\lambda)) = f((1 - \lambda)x(\theta^0) + \lambda x(\theta^1))$

\[\leq (1 - \lambda)f(x(\theta^0)) + \lambda f(x(\theta^1)) = (1 - \lambda)g(\theta^0) + \lambda g(\theta^1).\]

Therefore $C1 \iff C2$.

The proof that $SC1 \iff SC2$ is almost identical.

QED
Proposition: If $f$ is differentiable on $S$ then $C1 \Leftrightarrow C3$

Proof:

$C1 \Rightarrow C3$:

$C1 \Leftrightarrow C2$ therefore $g(\theta^z) \leq (1-\lambda)g(\theta^0) + \lambda g(\theta^0)$ for all $\lambda \in (0,1)$.

Rearranging the inequality:

\[ g(\theta^z) - g(\theta^0) \leq \lambda (g(\theta^1) - g(\theta^0)) \]

and so

\[ \frac{g(\theta^z) - g(\theta^0)}{\theta^z - \theta^0} \left( \frac{\theta^z - \theta^0}{\lambda} \right) \leq g(\theta^1) - g(\theta^0). \]

Since $\theta^z - \theta^0 = \lambda(\theta^1 - \theta^0)$ it follows that

\[ \frac{g(\theta^z) - g(\theta^0)}{\theta^z - \theta^0} (\theta^1 - \theta^0) \leq g(\theta^1) - g(\theta^0). \]
Taking the limit,

\[ g'(\theta^0)(\theta^1 - \theta^0) \leq g(\theta^1) - g(\theta^0) \, . \]

\( C3 \implies C1 \)

If \( x(\theta^0), x(\theta^1) \in S \) then by the convexity of \( S \) \( x(\theta^\lambda) \in S \).

Appealing to \( C3 \),

\[ g'(\theta^\lambda)(\theta^0 - \theta^\lambda) \leq g(\theta^0) - g(\theta^\lambda) \quad \text{and} \quad g'(\theta^\lambda)(\theta^1 - \theta^\lambda) \leq g(\theta^1) - g(\theta^\lambda) \]

Therefore

\[ g'(\theta^\lambda)((1 - \lambda)\theta^0 - (1 - \lambda)\theta^\lambda) \leq (1 - \lambda)g(\theta^0) - (1 - \lambda)g(\theta^\lambda) \]

and

\[ g'(\theta^\lambda)(\lambda\theta^1 - \lambda\theta^\lambda) \leq \lambda g(\theta^1) - \lambda g(\theta^\lambda) . \]

Adding the two inequalities yields

\[ 0 \leq (1 - \lambda)g(\theta^0) + \lambda g(\theta^1) - g(\theta^\lambda) . \]

QED
Proposition: if $f$ is differentiable on $S$ then $SC1 \iff SC3$

**Proof:**

$SC3 \Rightarrow SC1$:

The proof is essentially the same as the proof that $C3 \Rightarrow C1$.

$SC1 \Rightarrow SC3$:

By $SC1 \Rightarrow C1 \Rightarrow C3$. Therefore $g'(\theta^0)(\theta^1 - \theta^0) \leq g(\theta^1) - g(\theta^0)$.

But $\theta^1 - \theta^0 = \lambda(\theta^1 - \theta^0)$ and by $SC3$, $g(\theta^1) - g(\theta^0) < \lambda(g(\theta^1) - g(\theta^0))$.

Therefore

$$\lambda g'(\theta^0)(\theta^1 - \theta^0) < \lambda(g(\theta^1) - g(\theta^0))$$

QED
C4: For \( f \in C^1 \) and for any \( x^0, x^1 \in S \), define \( g(\theta) = f((1 - \theta)x^0 + \theta x^1) \)

\( x(\alpha), x(\beta) \in S \Rightarrow (\beta - \alpha)[g'(\beta) - g'(\alpha)] \geq 0 \)

Graphically, C4 is the statement that the slope of \( g(\lambda) \) is increasing.

**Proposition: If \( f \) is differentiable then \( C4 \iff C3 \)**

**Proof:**

\( C4 \Rightarrow C3 \)

\[
f(x^1) - f(x^0) = g(1) - g(0) = \int_0^1 g'(\lambda) d\lambda \geq \int_0^1 g'(0) d\lambda = g'(0) \int_0^1 d\lambda = g'(0)
\]

But \( g'(\lambda) = \frac{\partial f}{\partial x}(x^1). (x^1 - x^0) \).

Therefore \( f(x^1) - f(x^0) \geq g'(0) = \frac{\partial f}{\partial x}(x^0). (x^1 - x^0) \)
C3 ⇒ C4

**Proof:** Since \( f \) is convex, \( g \) is convex. Appealing to C3,

\[
g(\beta) \geq g(\alpha) + g'(\alpha)(\beta - \alpha) \quad \text{and} \quad g(\alpha) \geq g(\beta) + g'(\beta)(\alpha - \beta)
\]

Rewriting the second inequality,

\[
g(\beta) - g(\alpha) \leq g'(\beta)(\beta - \alpha)
\]

QED

An equivalent way of writing this is that for any \( \alpha \) and \( \beta \neq \alpha \)

\[
\frac{g'(\beta) - g'(\alpha)}{\beta - \alpha} \geq 0 \quad (*)
\]

That is, the second derivative of \( g(\lambda) \) must be positive.

Also if the second derivative is positive then (*) holds.

Therefore if \( f \in \mathbb{C}^2 \) C4 and C5 are also equivalent.

**C5:** For \( f \in \mathbb{C}^2 \) and any \( x^0, x^1 \in S \)

\[
g''(0) \geq 0 \quad \text{where} \quad g(\lambda) = f(x(\lambda)) = f((1 - \lambda)x^0 + \lambda x^1))
\]
Restatement of C5

Suppose that $f$ is twice differentiable on $\mathbb{R}^n$

Define $g(\lambda) = f(x^\lambda) = f(x^0 + \lambda(x^1 - x^0))$.

$$
\frac{d^2 g}{d\lambda^2}(0) = \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i^1 - x_i^0) \frac{\partial^2 f}{\partial x_i \partial x_j}(x_j^1 - x_j^0) = (x^1 - x^0)' \left[ \frac{\partial^2 f}{\partial x_i \partial x_j}(x^0) \right] (x^1 - x^0) \geq 0
$$

Then we can rewrite C5 as follows:

C5: For $f \in \mathbb{C}^2$ defined on $S$ and any $x^0, x^1 \in S$

$$(x^1 - x^0)' H(x^0)(x^1 - x^0) \geq 0 \text{ where } H(x^0) = \left[ \frac{\partial^2 f}{\partial x_i \partial x_j}(x^0) \right]$$

If $S = \mathbb{R}^n$ then C5 is the statement that the “Hessian Matrix” $H(x^0)$ must be positive semi-definite for all $x^0 \in \mathbb{R}^n$
**SC4:** For $f \in C^1$ and for any $x^0, x^1 \in S$, define $g(\theta) = f((1 - \theta)x^0 + \theta x^1)$

$x(\alpha), x(\beta) \in S \Rightarrow (\beta - \alpha)[g'(\beta) - g'(\alpha)] > 0$

Exercise: Prove that SC4 $\Rightarrow$ SC3

HINT: Proof follows the proof that C4 $\Rightarrow$ C3

Exercise: Prove that SC3 $\Rightarrow$ SC4

HINT: Appeal to the definition first with $\theta^0 = \alpha$ then with $\theta^0 = \beta$. 
Strictly Concave Function

$f$ is a strictly concave function if, for any $x^0, x^1 \in S$ and convex combination $x^\lambda$, $0 < \lambda < 1$,

$$f(x^\lambda) < (1 - \lambda)f(x^0) + \lambda f(x^1)$$

Concave Function

$f$ is a concave function if, for any $x^0, x^1 \in S$ and convex combination $x^\lambda$, $0 < \lambda < 1$,

$$f(x^\lambda) \leq (1 - \lambda)f(x^0) + \lambda f(x^1)$$

Reverse all the inequalities in $C1 - C5$ and $SC1 - SC4$ to obtain equivalent definitions of a concave and strictly concave function.

**Exercise:** Show that a function $f$ is concave on $S$ if and only if $-f$ is convex on $S$. 
Class exercises (in groups)

1. Prove that a concave function of a concave function is not necessarily concave.

2. Prove that a strictly concave function of a strictly concave function maybe strictly convex.

3. Show that the sum of concave functions is concave.

4. Show that the sum of strictly concave function is strictly concave.

5. “If \( f_1(x_1, x_2) \) is strictly concave and \( f_1(x_3, x_4) \) is concave then \( h(x) = f(x_1, x_2) + g(x_3, x_4) \) is strictly concave.”

   Either prove that this statement is true or present a counter example establishing that the statement is false.
Shortcuts for checking whether a function is convex or concave

With one variable, the easiest way to tell if a function is strictly convex is to see if it has a positive second derivative. For functions of more than one variable, the following propositions are often helpful.

**Proposition: The sum of concave functions is convex.**

**Proposition: Concave function of a function**

$h(x) = g(f(x))$ is concave if (i) $f$ is concave & $g$ is increasing and concave, or (ii) $f$ is linear and $g$ is concave.

Proof of (i)

Since $f$ is concave, \[ f(x^\lambda) \geq (1 - \lambda)f(x^0) + \lambda f(x^1). \]

Since $g$ is increasing \[ g(f(x^\lambda)) \geq g((1 - \lambda)f(x^0) + \lambda f(x^1)) \]

Define \[ y^0 = f(x^0) \quad y^1 = f(x^1) \]

then \[ g(f(x^\lambda)) \geq g((1 - \lambda)y^0 + \lambda y^1) \quad (\star) \]

Since $g$ is convex \[ g((1 - \lambda)y^0 + \lambda y^1) \geq (1 - \lambda)g(y^0) + \lambda g(y^1) = (1 - \lambda)g(f(x^0)) + \lambda g(f(x^1)). \]

Appealing to (\star) \[ g(f(x^\lambda)) \geq (1 - \lambda)g(f(x^0)) + \lambda g(f(x^1)) \]
Quasi-concave Function

A function \( f \) is quasi-concave on the convex set \( S \subseteq \mathbb{R}^n \) if, for any \( x^0, x^1 \) in S, such that \( f(x^1) \geq f(x^0) \) and any convex combination \( x^\lambda = (1-\lambda)x^0 + \lambda x^1, \ 0 < \lambda < 1 \),

\[
    f(x^\lambda) \geq f(x^0).
\]

Strictly Quasi-concave Function

A function \( f \) is quasi-concave on the convex set \( S \subseteq \mathbb{R}^n \) if, for any \( x^0, x^1 \) in S, such that \( f(x^1) \geq f(x^0) \) and any convex combination \( x^\lambda = (1-\lambda)x^0 + \lambda x^1, \ 0 < \lambda < 1 \),

\[
    f(x^\lambda) > f(x^0).
\]

Example: \( U(x) = x_1 x_2 \) is quasi-concave on \( \mathbb{R}^2_+ \) and is strictly quasi-concave on

\[
    \mathbb{R}^2_{++} = \{ x | x \in \mathbb{R}^2, \ x >> 0 \}
\]

But why! Read on.
Proposition: A function is quasi-concave if and only if the upper contour sets of the function are convex.

Proof of equivalence: Without loss of generality we may assume that \( f(x^1) \geq f(x^0) \).

Suppose \( f \) is quasi-concave and for any \( \hat{x} \), consider vectors \( x^0 \) and \( x^1 \) which lie in the upper contour set \( C_U(\hat{x}) \) of this function. That is \( f(x^0) \geq f(\hat{x}) \) and \( f(x^1) \geq f(\hat{x}) \). From the definition of quasi-concavity, for any convex combination,

\[
x^\lambda = (1 - \lambda)x^0 + \lambda x^1, \quad f(x^\lambda) \geq f(x^0).
\]

Since \( f(x^0) \geq f(\hat{x}) \), it follows that for all convex combinations \( f(x^\lambda) \geq f(\hat{x}) \). Hence all convex combinations lie in the upper contour set. Thus \( C_U(\hat{x}) \) is convex.

Conversely, if the upper contour sets are convex, then \( C_U(x^0) \) is convex. Since \( f(x^1) \geq f(x^0) \). Then \( x^0 \) and \( x^1 \) are both in \( C_U(x^0) \) therefore, by the convexity of \( C_U(x^0) \), all convex combination lie in this set. That is \( f(x^\lambda) \geq f(x^0) \) for all \( \lambda, \ 0 < \lambda < 1 \).

Q.E.D.
**Proposition:** If there is some strictly increasing function $g$ such that $h(x) \equiv g(f(x))$ is concave then $f(x)$ is quasi-concave.

**Proof:** We need to establish that if $f(x^1) \geq f(x^0)$ then $f(x^\lambda) \geq f(x^0)$, $0 < \lambda < 1$

By concavity $h(x^\lambda) \geq (1-\lambda)h(x^0) + \lambda h(x^1)$, that is $g(f(x^\lambda)) \geq (1-\lambda)g(f(x^0)) + \lambda g(f(x^1))$. (*)

Since $g$ is increasing and $f(x^1) \geq f(x^0)$, $g(f(x^1)) \geq g(f(x^0))$. Then substituting into (*),

$g(f(x^\lambda)) \geq g(f(x^0))$.

By hypothesis, $g$ is a strictly increasing function. Then $f(x^\lambda) \geq f(x^0)$. Q.E.D.

**Remark:** It is in this last line of the proof that we require that $g$ is strictly increasing. For otherwise $g$ could be constant over some neighborhood $N(x^0, \delta)$ and so we could have $f(x^1) < f(x^0)$ and $g(f(x^1)) = g(f(x^0))$. 
Proposition B.2-9: If \( f : \mathbb{R}^n_+ \to \mathbb{R} \) is positive, homogeneous of degree 1 (i.e. for all \( \lambda > 0 \) \( f(\lambda z) = \lambda f(z) \)) and quasi-concave then \( f \) is concave.

Proof: Because \( f \) is homogeneous of degree 1 it follows that for any \( x^0, x^1 \) and \( \lambda \in (0,1) \)

\[
(1 - \lambda) f(x^0) + \lambda f(x^1) = f((1 - \lambda)x^0) + f(\lambda x^1),
\]

Since \( f > 0 \), there exists \( \theta > 0 \) such that

\[
f((1 - \lambda)x^0) = \theta f(\lambda x^1).
\]

Then

\[
(1 - \lambda) f(x^0) + \lambda f(x^1) = (1 + \theta) f(\lambda x^1).
\]

Because \( f \) is homogeneous of degree 1 it follows from (1) that \( f((1 - \lambda)x^0) = f(\theta \lambda x^1) \). Note that

\[
\frac{\theta}{1 + \theta}((1 - \lambda)x^0 + \lambda x^1) = \frac{\theta}{1 + \theta}(1 - \lambda)x^0 + \frac{1}{1 + \theta}\theta \lambda x^1.
\]

Therefore \( \frac{\theta}{1 + \theta}((1 - \lambda)x^0 + \lambda x^1) \) is a convex combination of \( (1 - \lambda)x^0 \) and \( \theta \lambda x^1 \).
We showed that \( \frac{\theta}{1+\theta}((1-\lambda)x^0 + \lambda x^1) \) is a convex combination of \((1-\lambda)x^0 \) and \(\theta \lambda x^1\).

Therefore, by the quasi-concavity of \(f\)

\[
f\left(\frac{\theta}{1+\theta}((1-\lambda)x^0 + \lambda x^0)\right) \geq f(\theta \lambda x^1).
\]  

(3)

Because \(f\) is homogeneous of degree 1 it follows that

\[
\frac{\theta}{1+\theta} f((1-\lambda)x^0 + \lambda x^0) \geq \theta f(\lambda x^1)
\]

and hence that

\[
f((1-\lambda)x^0 + \lambda x^0) \geq (1+\theta) f(\lambda x^1).
\]

Appealing to (2),

\[
f((1-\lambda)x^0 + \lambda x^0) \geq (1-\lambda) f(x^0) + \lambda f(x^1).
\]

QED
Class examples:

In each case is $f$ (i) quasi-concave (ii) strictly quasi-concave (iii) concave

(a) \[ f(x) = (1 + x_1)(2 + x_2), \ x \in \mathbb{R}^2_+ \]
(b) \[ f(x) = x_1x_2, \ x \in \mathbb{R}^2_+ \]
(c) \[ f(x) = x_1x_2, \ x \in \mathbb{R}^{2++} \]
(d) \[ f(x) = x_1x_2 + x_1x_3, \ x \in \mathbb{R}^3_{++,+} \]
(e) \[ f(x) = x_1^{1/2} + x_2^{1/2} + x_3^{1/2} \]
(f) \[ f(x) = (x_1^{1/2} + x_2^{1/2} + x_3^{1/2})^2 \]
(g) \[ f(x) = (x_1^{1/2} + x_2^{1/2} + x_3^{1/2})^{3/2} \]
**Class Exercise:** Are the following statements true or false?

1. For any convex set $S \subseteq \mathbb{R}^n$ if the quasi-concave function $f : S \rightarrow \mathbb{R}$ has a local maximum at $x^*$, then $f$ takes on its global maximum at $x^*$.

2. For any set $S \subseteq \mathbb{R}^n$ if the strictly quasi-concave function $f : S \rightarrow \mathbb{R}$ has a local maximum at $x^*$, then $f$ takes on its global maximum at $x^*$. 
Supporting hyperplane for a convex set

Suppose that for some $x^0$, the upper contour set

$$S = \{x \mid f(x) \geq f(x^0)\}$$

is convex.

(This will be true if $f$ is quasi-concave.)

The linear approximation of $f$ at $x^0$ is

$$l(x) = f(x^0) + \frac{\partial f}{\partial x}(x^0) \cdot (x - x^0).$$

Consider the upper contour set of $l$. This is the set

$$T = \{x \mid l(x) \leq l(x^0)\} = \{x \mid \frac{\partial f}{\partial x}(x^0) \cdot (x - x^0) \geq 0\}$$

We will argue that $S \subseteq T$. 
Proposition: Supporting hyperplane

If the set \( S = \{x \mid f(x) \geq f(x^0)\} \) is convex, then \( S \subset T \equiv \{x \mid \frac{\partial f}{\partial x}(x^0) \cdot (x - x^0) \geq 0\} \).

The hyperplane \( \frac{\partial f}{\partial x}(x^0) \cdot (x - x^0) = 0 \) touching the boundary of \( S \) at \( x^0 \) is called a supporting hyperplane.

Proof: For \( x^1 \in S \) define \( g(\lambda) = f(x(\lambda)) - f(x^0) = f(x^0 + \lambda(x^1 - x^0)) \).

Since \( S \) is convex, \( x(\lambda) \) is in \( S \). Therefore \( g(\lambda) \geq 0 \). Since \( g(0) = 0 \) it follows that \( \frac{dg}{d\lambda}(0) \geq 0 \).

But as we have previously shown,

\[
\frac{dg}{d\lambda}(0) = \frac{\partial f}{\partial x}(x^0) \cdot (x^1 - x^0).
\]

Then for all \( x^1 \in S \), \( \frac{\partial f}{\partial x}(x^0) \cdot (x^1 - x^0) \geq 0 \). Q.E.D.
**Example 1:**

Consider the contour set of $f(x) = (x_1 + 2x_2^{1/2})^2$ through $(1,1)$.

The upper contour set is

$$S = \{x \mid (x_1 + 2x_2^{1/2})^2 \geq 9\}$$

Confirm that $f(x) = (x_1 + 2x_2^{1/2})^2$ is quasi-concave.

The linear approximation of $f$ at $x^0$ is

$$l(x) = f(x^0) + \frac{\partial f}{\partial x}(x^0) \cdot (x - x^0) = 9 + 6(x_1 - 1) + 6(x_2 - 1).$$

Consider the upper contour set of $l$. This is the set

$$T = \{x \mid l(x) \geq l(x^0)\} = \{x \mid 6(x_1 - x_1^0) + 6(x_2 - x_2^0) \geq 0\}$$

**Example 2: Production set of a firm**

$Y = \{(z,q) \mid 3z^\alpha - q \geq 0\}$ where $\alpha \in (0,1]$ . Consider the tangent line through $(1,3)$
Application: Market clearing in an exchange economy

Suppose all consumers have the same strictly increasing quasi-concave utility function $U(\cdot) \in \mathbb{C}^1$ defined on the $\mathbb{R}^n$. Consumer $h$ has endowment vector $\omega^h$. Finally suppose that

$$\frac{\partial U}{\partial x}(\omega^h) \cdot \omega^h > 0$$

Appealing to the Supporting hyperplane proposition,

$$U(x^h) \geq U(\omega^h) \Rightarrow \frac{\partial U}{\partial x}(\omega^h) \cdot (x^h - \omega^h) \geq 0$$

Define the price vector $p = \frac{\partial U}{\partial x}(\omega^h)$. Since $U$ is strictly increasing $p \geq 0$. Then

$$U(x^h) \geq U(\omega^h) \Rightarrow p \cdot (x^h - \omega^h) \geq 0 \quad (*)$$

Thus any strictly preferred consumption bundle $x^1$ costs at least as much as $\omega^0$. We wish to show that

$$U(x^h) > U(\omega^h) \Rightarrow p \cdot (x^h - \omega^h) > 0. \quad (**)$$

That is, any strictly preferred bundle costs strictly more. Then the consumer can do no better than consume his or her endowment.
Suppose (***) is false.

Then for some $\hat{x}$, $U(\hat{x}) > U(\omega^h)$ and $p \cdot \hat{x} = p \cdot \omega^h$.

By hypothesis, $\frac{\partial U}{\partial x}(\omega^h) \cdot \omega^h > 0$. Hence $p \cdot \omega^h > 0$.

Then for all $x^\lambda = \lambda \hat{x}$, $0 < \lambda < 1$, $0 < p \cdot x^\lambda < p \cdot \omega^0$.

Since $U(\hat{x}) > U(\omega^h)$ and $U$ is continuous, it follows that $U(x^\lambda) > U(\omega^h)$ for all $\lambda$ sufficiently close to 1.

Summarizing our conclusions:

For $\lambda$ sufficiently close to 1, $U(x^\lambda) > U(\omega^h)$ and $p \cdot x^\lambda < p \cdot \omega^h$.

But this contradicts (*).

Thus there can be no such $\hat{x}$.

QED

We have therefore shown that if the price vector is $p$, then consumer $h$ will not gain from trading. Then if all consumers have the same endowment, none will gain from trading. Thus $p$ is a market clearing price vector.