# Price Discrimination Lectures

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1. Direct Price Discrimination

Figure 1.1: Demand price and marginal revenue
How much a monopoly charges above marginal cost depends on the demand elasticity.

\[ MR_t = \frac{d}{dq} q p_t(q) = p_t(q) + q \frac{dp_t}{dq} = p_t(1 + \frac{q}{p_t} \frac{dp_t}{dq}) = p_t(1 - \frac{1}{\varepsilon_t}). \]

Since the monopoly equates MR and MC,

\[ p_t(1 - \frac{1}{\varepsilon_t}) = MC, \text{ hence } p_t = \frac{MC}{1 - \frac{1}{\varepsilon_t}}. \]

Differences in price elasticity lead to different prices. The more negative the elasticity of the demand function the lower is the price.
2. Direct Price Discrimination with two part pricing

Fixed access fee $K_t$ (per month, for example)

use fee (or price) $p_t$.

A type $t$ customer has demand $q_t(p)$.

Linear case with demand price function $p = a_t - b_t q$

$$q_t(p) = \max(0, (a_t - p) / b_t).$$

WHY?

Total payment from the use fee $= p_t q_t(p_t)$.

net revenue to the firm $= p_t q_t(p_t) - c q_t(p_t)$.

This is the yellow area.

Figure 2.1: Demand price and consumer surplus
The total benefit to the consumer $B_t(q)$ is the area under the demand price function. Subtracting off the use costs yields the net gain or “consumer surplus” $CS_t(p_t)$ This is the pink triangle.

In the case of linear demand price functions the consumer surplus is half the rectangle, that is

$$CS_t(p_t) = \frac{1}{2}(a_t - p_t)q_t(p_t)$$

This is entered in Solver in cell G6. (see the formula bar.)

If the consumer pays an access fee of $K_t$ her net gain is $u_t(p_t, K_t) = CS_t(p_t) - K_t$

This is entered in Solver as shown below in cell J6.
Maximizing Profit

\[ u_t(p_t, K_t) = CS_t(p_t) - K_t \]

1. For any price \( p_t \), raise the access fee \( K_t \) till the consumer is ready to exit from the market.
2. Total profit is the sum of the two shaded areas. Note that this is social surplus.
3. Lowering \( p_t \) raises social surplus = profit as long as \( p_t > c \). Therefore set \( p_t = c \) and extract the social surplus with the access fee.

Note that the monopolist no longer under supplies.

The demand price \( p_t(c) = MB = MC \)

\[\text{Figure 2.1: Demand price and consumer surplus}\]
An alternative approach

With the two part plan \((\hat{p}, \hat{K})\), consumer surplus is

The area to the left of the demand price function.

\[
CS_t(\hat{p}) = \int_{\hat{p}}^{a_t} q_t(x)dx.
\]

The buyer’s payoff for a plan \((p, K)\) is therefore

\[
U_t(p, K) = CS_t(p) - K
\]

The buyer has the option of purchasing nothing.

Equivalent to a plan \((p_0, K_0)\) with a price \(p_0 \geq a_t\) and an access fee \(K_0 = 0\).

The profit of the monopolist is

\[
\Pi(p, K) = (p - c)q_t(p) + K
\]
Indifference curve for a customer

\[ U_i(p, K) = CS_i(p) - K = U_i(\hat{p}, \hat{K}) \]

Rearranging,

\[ K = CS_i(p) - U_i(\hat{p}, \hat{K}) = \int_p^{a_i} q_i(x)dx - U_i(\hat{p}, \hat{K}) \]

The slope of the indifference curve is

\[ \frac{dK}{dp} = -q_i(p) \]
Indifference curve for the monopoly

\[ \Pi_t(p, K) = (p - c)q_t(p) + K = \Pi_t(\hat{p}, \hat{K}) \]

Rearranging,

\[ K = \Pi_t(\hat{p}, \hat{K}) - (p - c)q_t(p) \]

The slope of the indifference curve is

\[ \frac{dK}{dp} = -q_t(p) - (p - c) \frac{dq_t}{dp} . \]

**Note that both have the same slope at** \( p = c \)
Profit maximizing two part pricing

\[ U_i(p, K) \geq U_i(c, K^*) \]

\[ \Pi_i(p, K) = \Pi_i(c, K^*) \]
3. **Indirect price discrimination with two part pricing**

For perfect direct price discrimination we need assumptions

(i) resale is prohibitively costly  
(ii) the monopoly can identify the different types of buyer  
(iii) there are no legal constraints to excluding different classes of customer from offers.

Henceforth we consider environments where either (ii) or (iii) is false.

**Indirect price discrimination**

No exclusion but plans designed to attract different types of customer.

*T different types of customer.*

Assumption 1: Higher types have higher demand price functions

For all \( s \) and \( t > s \) if \( p_s(q) \geq 0 \) then \( p_t(q) > p_s(q) \)
For simplicity linear demand price functions. \( p_t = a_t - b_t q, \quad t = 1, \ldots, T \)

The monopoly offers a set of alternatives or “plans”. \( \{(p_1, K_1), \ldots, (p_n, K_n)\} \).

Each consumer picks one of these alternatives or purchases nothing.

The alternative \( (p_0, K_0) = (a_T, 0) \) is equivalent to purchasing nothing.

add this to the set

\( \{(p_0, K_0), (p_1, K_1), \ldots, (p_n, K_n)\} \)
Define \((p_s, K_s)\) to be the choice of type \(s\) from the set.

\[\{(p_0, K_0), (p_1, K_1), \ldots, (p_T, K_T)\}\]

Since \((p_s, K_s)\) is the choice of a type \(s\) buyer,

\[u_s(p_s, K_s) \geq u_s(p_t, K_t), \text{ for all } (p_t, K_t) \text{ in } C,\]

where

\[u_s(p_t, K_t) = CS_s(p_t) - K_t\]
We now derive two simple but important results.

**Principle 1:** Let \((p_s, K_s)\) be the choice of a type \(s\) buyer and let \((p_t, K_t)\) be the choice of a higher type \(t\) (so that \(p_t(q) > p_s(q)\)). Then \(p_t \leq p_s\): the higher demander chooses a plan with a lower use fee.

Let \((p_s, K_s)\) be the choice of type \(s\) and let \((p_t, K_t)\) be the choice of type \(t\). We suppose \(p_t > p_s\) and show that this is impossible.

Since type \(t\) prefers \((p_t, K_t)\)

\[ U_t(p_t, K_t) \geq U_t(p_s, K_s) \]

\[ CS_t(p_t) - K_t \geq CS_t(p_s) - K_s \]

Equivalently

\[ CS_t(p_s) - CS_t(p_t) \leq K_s - K_t \]

The gain in consumer surplus in switching to the lower priced plan is less than or equal to the increase in the access fee \(K_s - K_t\).

In the figure, the gain in consumer surplus for a type \(t\) customer is the sum of the shaded and dotted areas. So shaded area \(\leq K_s - K_t\)
In the figure, the gain in consumer surplus for a type $t$ customer is the sum of the shaded and dotted areas. So shaded area $\leq K_s - K_t$

For any lower type the gain in consumer surplus (the shaded area) is smaller so this customer is strictly worse off switching to plan $s$.

But then $(p_s, K_s)$ cannot be the choice of a type $s$ customer.
**Principle 2:**

If type $s$ is indifferent between 2 plans $(p_a, K_a)$ and $(p_b, K_b)$ and $p_a > p_b$ then (i) all higher types strictly prefer the plan with the lower use fee $(p_b, K_b)$ and (ii) all lower types strictly prefer the plan with the higher use fee $(p_a, K_a)$.

We have already established (i) in the discussion of Principle 1. To see that (ii) is also true we proceed in essentially the same fashion.

Since type $s$ is indifferent between the two plans the gain in consumer surplus in switching from plan $a$ to plan $b$ must be equal to the increase in the access fee $K_b - K_a$. For a lower type the increase in consumer surplus is lower.
Special case: Two types

Suppose that type 1 buyers choose \((p_1, K_1)\) and type 2 buyers choose \((p_2, K_2)\).

Payoffs:

\[
\begin{align*}
u_1(p_1, K_1) &= CS_1(p_1) - K_1 \\
u_2(p_2, K_2) &= CS_2(p_2) - K_2
\end{align*}
\]

Suppose that a type 1 buyer’s payoff is strictly positive. Then we can raise the access fee by an equal amount \(\Delta K\) on both plans until type 1 buyers have a payoff of (almost) zero. Since the cost of both plans has gone up by the same amount, type 2 buyers will not switch plans.

Hence we have the following result.

**Principle 3a:** The profit maximizing monopolist chooses \(K_1 = CS_1(p_1)\) so that the payoff of type 1 is zero.
Note next that if a type 2 buyer chooses plan 1 her payoff is \( U_2(p_1, K_1) = CS_2(p_1) - K_1 \)

If she choose plan 2 her payoff is \( U_2(p_2, K_2) = CS_2(p_2) - K_2 \).

Therefore the access fee for a type 2 customer can be raised until she is (almost) indifferent between the two plans.

**Principle 3b:** The monopolist chooses \( K_2 \) so that a type 2 customer is indifferent between plan 2 and plan 1, that is \( U_2(p_2, K_2) = CS_2(p_2) - K_2 = U_2(p_1, K_1) \)

Marginal payoff

\[
U_2(p_2, K_2) = CS_2(p_2) - K_2 = U_2(p_1, K_1) = CS_2(p_1) - K_1
\]

From Principle 3a \( CS_1(p_1) - K_1 = 0 \).

\[
U_2(p_2, K_2) = CS_2(p_2) - K_2 = CS_2(p_1) - K_1 = CS_2(p_1) - CS_1(p_1) + [CS_1(p_1) - K_1]
\]

\[
= CS_2(p_1) - CS_1(p_1) \equiv M_2
\]

Note that \( M_2 \) is the extra consumer surplus that a type 2 buyer gets by choosing plan 1.

The monopoly cannot extract this surplus. Then \( K_2 = CS_2(p_2) - M_2 \)
Turn to SOLVER
More than 2 types of buyer

Review: Two types: plans \((p_0, K_0), (p_1, K_1), (p_2, K_2)\) where plan 0 is \((p_0, 0)\), equivalent to buying nothing if \(p_0\) is sufficiently large.

By Principle 1, \(p_0 \geq p_1 \geq p_2\)

By Principle 3: \(U_1(p_1, K_1) = U_1(p_0, K_0)\) and \(U_2(p_2, K_2) = U_2(p_1, K_1)\)

We can use Solver to compute these payoffs and then impose these equality constraints.

With more than 2 types we proceed in the same way, choose \(p_0 \geq p_1 \geq \ldots \geq p_T\) and impose the “local downward constraints”

\[U_t(p_t, K_t) = U_t(p_{t-1}, K_{t-1})\]

But how do we know that a buyer may not wish to switch some other plan?
Consider the 3 type case: \( u_{ij} \equiv U_t(p_j, K_j) \) payoff to type \( t \) if she chooses plan \( j \)

| Type \( t \) | Plan \( j \) |
|-----|-----|-----|-----|-----|-----|
| 1   | \( u_{10} \) = | \( u_{11} \) | ?   | \( u_{12} \) | ?   | \( u_{13} \) |
| 2   | \( u_{20} \) ? | \( u_{21} \) = | \( u_{22} \) | ?   | \( u_{23} \) |
| 3   | \( u_{30} \) ? | \( u_{31} \) ? | \( u_{32} \) = | \( u_{33} \) |

We now argue that below the red borders all the question marks should be replaced by \( \leq \) and above the green borders all the question marks should be replaced by \( \geq \).
Below the red borders

Consider the first column. If they are the same plans all other types will also be indifferent. Then suppose the plans are different. By Principle 1, \( p_0 > p_1 \). By Principle 2, since a type 1 buyer is indifferent between plan 1 and plan 0, all higher types strictly prefer plan 1. Exactly the same argument holds for column 2.

<table>
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<th>2</th>
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<td>( u_{10} ) = ( u_{11} )</td>
<td>?</td>
<td>( u_{12} )</td>
<td>?</td>
</tr>
<tr>
<td>2</td>
<td>( u_{20} ) ( \leq ) ( u_{21} ) = ( u_{22} )</td>
<td>?</td>
<td>( u_{23} )</td>
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<tr>
<td>3</td>
<td>( u_{30} ) ( \leq ) ( u_{31} ) ( \leq ) ( u_{32} ) = ( u_{33} )</td>
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Above the green borders

Consider the third column. If plan 2 and plan 3 are the same plan all other types will be indifferent. Then suppose the plans are different. By Principle 1, $p_2 > p_3$. By Principle 2, since a type 3 buyer is indifferent between plan 3 and plan 2, all lower types strictly prefer plan 2. Exactly the same argument holds for column 2.

<table>
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<td>u30</td>
<td>≤</td>
<td>u31</td>
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</table>
The same argument holds for any number of types. Hence we have the following further principle.

**Principle 4:**

If each of the local downward constraints $U_t(p_t, K_t) \geq U_t(p_{t-1}, K_{t-1})$ is an equality constraint, then no buyer type gains by switching to any other plan.

**Class Example:** Study the effect of changing all the parameters except the cost parameter.
Intuition

Consider the left figure.

Step 1:
Set use fees \( p_1 \) and \( p_2 \leq p_1 \)

Consumer surplus for low demander is the red dotted area.

Step 2:
Choose access fee \( K_1 \) (dotted area) to appropriate all the consumer surplus for low demanders.

Step 3:
Determine how big a payoff each high demander can get by switching to plan 1 (green striped area.)

Consider the right figure. The high demander’s consumer surplus is the sum of the pink dotted and green striped areas. The monopoly can raise the access fee \( K_2 \) until the high demander is (almost) indifferent between the two plans. Then the profit maximizing access fee is the dotted pink area.
Choosing $p_2$

The monopoly gets the social surplus on the high demanders less the payoff if the high demander switches to plan 1. Thus the monopoly wants the social surplus to be as big as possible. Then $p_2 = c$. 

\[ p_1 = \frac{a_2 - a_1}{q_1(p_1)} \]

\[ p_2 = \frac{a_2 - a_1}{q_2(p_2)} \]
The final step is to ask what happens as $p_1$ declines to $\hat{p}_1$.

**Choosing $p_1$**

The monopoly appropriates the entire social surplus on plan 1 so when $p_1$ declines the increased profit is the area bounded by the heavy lines.

The monopoly also appropriates all the social surplus on each high demander less the high demander’s payoff to switching plans. Note that the payoff to switching has risen by the area bounded by the heavy lines in the right hand figure. The monopoly must reduce the access fee $K_2$ by this amount to stop a high demander from switching plans $\Delta \Pi = n_1 \Delta S_1 - n_2 \Delta K_2$.

**Class exercise:** Why does the monopoly choose $p_1 > c$?
“Cell phone plans”

Rather than offering a plan in which the consumer chooses the number of units, suppose that the firm offers a fixed number of units and a total payment per month.

The firm offers a set of alternative plans

\{ (q_a, r_a), ..., (q_n, r_n) \}

A customer has the option of not participating. In this case the outcome is \((q_0, r_0) = (0, 0)\). We will call this “plan 0”. Then the set of alternatives available to a customer is

\{ (q_0, r_0), (q_a, r_a), ..., (q_n, r_n) \}

Let \((q_s, r_s)\) be the choice of a type \(s\) customer and define

\{ (q_0, r_0), (q_1, r_1), ..., (q_T, r_T) \}

to be the set of choices of all the different types.
“Cell phone” plans are more profitable

Consider the lowest plan with 2 part pricing. The monopoly extracts all the consumer surplus from type 1 by charging an access fee $K_1$ equal to $CS_1(p_1)$.

Now consider a type 2 customer who switches a purchases plan 1. Her consumer surplus is the sum of the dotted and striped areas. She also has to pay the access fee $K_1$. However, as we have just argued, the dotted area is equal to $K_1$. Then the payoff to a type 2 buyer if she switches is the blue striped area.

Fig. 4.2: Payoff to a type 2 buyer with 2 part pricing
Note that the total payment by type 1 is the sum of the shaded and dotted areas.

\[ r_1 = p_1 * q_1(p_1) + K_1. \]

Suppose that the monopoly replaces plan 1 with a “cell phone” plan that offers \( q_1 = q_1(p_1) \) units for a monthly payment of \( r_1 \).

Exactly the same outcome for a type 1 buyer and the monopoly.

The consumer surplus of a type 2 buyer is the sum of the green striped and dotted areas. Since he must pay the sum of the dotted and shaded areas his payoff is the green striped area.

Compare with the striped area under 2 part pricing.

**Conclusion: switching is less attractive. So a type 2 buyer can be charged more for his plan.**
Two simple but important results.

**Proposition 1’**: Let \((q_s, r_s)\) be the choice of a type \(s\) buyer and let \((q_t, r_t)\) be the choice of a type \(t\) buyer where \(p_t(q) > p_s(q)\). Then \(q_s \leq q_t\).

To see that statement is true, let \((q_s, r_s)\) be the plan chosen by type \(s\) and let \((q, r)\) be a plan with a lower quantity.

For type \(s\) the loss in benefit from the lower quantity is the area under the demand price function. This must be at least equal to the reduction in payment \(r_s - r\).

For higher types the loss from switching to the smaller plan is greater. Therefore all higher types strictly prefer \((q_s, r_s)\) over \((q, r)\).
Proposition 2’:
If type \( s \) is indifferent between 2 plans \((q_a, r_a)\) and \((q_b, r_b)\) and \( q_a < q_b \) then (i) all higher types strictly prefer the plan with the higher quantity \((q_b, r_b)\) and (ii) all lower types strictly prefer the plan with the lower quantity \((q_a, r_a)\).

Statement (i) follows directly from the argument used to derive Principle 1’. To see that (ii) is true note that if a type \( s \) buyer switches from plan \( a \) to plan \( b \) his gain in benefit is the sum of the shaded regions. Since he is indifferent this must be just offset by the cost difference \( r_b - r_a \).

Any lower type has a lower gain and so his gain to switching is more than offset by the additional cost. Then lower types are strictly worse off purchasing plan \( b \).
It follows that we can proceed exactly as with two part pricing.

Choose some increasing quantity array \( \{q_1, ..., q_T\} \) and then choose total payments so that the local downward constraints are binding.

\[
\begin{align*}
    u_1(q_1, R_1) &= u_1(q_0, R_0) = u_1(0, 0) = 0 \\
    u_2(q_2, R_2) &= u_2(q_1, R_1)
\end{align*}
\]

etc.

Appealing to Principle 2’ we can confirm that all the incentive and participation constraints are binding.

Exercise: Look at the argument for two part pricing to see why this is true.
We then use SOLVER to choose the quantities that maximize the total profit of the firm.

1. Choose plan quantities where \( q_1 \leq q_2 \leq \ldots \leq q_T \)

2. Compute benefits

3. Compute benefits if switch to the closest smaller plan

4. Compute the added payoff that type \( t \) gets if he jumps down

where \( M_t = B_t(q_{t-1}) - B_{t-1}(q_{t-1}) \)

5. Compute buyer payoffs

For profit maximization, type \( t \) must be indifferent between his plan and the next smaller plan

\[
U_t(q_t, r_t) = U_t(q_{t-1}, r_{t-1}) \\
= B_t(q_{t-1}) - r_{t-1} \\
= B_t(q_{t-1}) - B_{t-1}(q_{t-1}) + B_{t-1}(q_{t-1}) - r_{t-1} \\
= M_t + U_{t-1}(q_{t-1}, r_{t-1})
\]

Note that lowest type has a payoff \( U_1(q_1, r_1) = B_1(q_1) - r_1 = 0 \).

6. Compute plan payments \( U_t(q_t, r_t) = B_t(q_t) - r_t \). Therefore \( r_t = B_t(q_t) - U_t(q_t, r_t) \)
INTUITION FOR THE 2 TYPE CASE

**Optimal choice of** $q_2$

The payoff for type 2 is

$$u_2(q_2, r_2) = B_2(q_2) - r_2.$$  

The benefit $B_2(q_2)$ is the pink area in the left diagram. If $q_2$ is increased the total benefit rises by the dark pink area.

Therefore the payment for the plan $\Delta r_1$ can be increased by this amount. In the right diagram the increase in profit is the green area. Thus profit is maximized by choosing a plan quantity so that the demand price $p_2(q_2) = c$. 

![Diagram showing the optimal choice of $q_2$ and the increase in profit due to an increase in $q_2$.](image)
**Optimal choice of \( q_1 \)**

Arguing as above, increasing the units for plan 1 by \( \Delta q \) increases profit on plan 1 as depicted in the left diagram below.

The extra benefit to type 1 is the dark pink area and so the monopoly raises the total payment for the plan by this amount. However the extra benefit of the new larger plan 1 to type 2 also includes the red area. Therefore the monopoly must **lower** the payment for plan 2 by \( \Delta r_2 \) to eliminate the incentive to switch.

\[
\Delta \Pi = n_1 \Delta \Pi_1 - n_2 \Delta r_2
\]